

# Weak Critical Sets in Cyclic Latin Squares

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## Abstract

We identify a weak critical set in each cyclic latin square of order greater than 5. This provides the first example of an infinite family of weak critical sets. The proof uses several constructions for latin interchanges which are generalisations of those introduced by Donovan and Cooper.

## 1 Introduction

A *latin square* of order  $n$  is an  $n \times n$  array with entries chosen from a set  $N$  of size  $n$  such that each element of  $N$  occurs exactly once in each row and column. We shall use  $N = \{0, 1, \dots, n - 1\}$  and label the rows and columns from 0 to  $n - 1$ . We may also represent a latin square by the set of  $n^2$  triples  $(i, j, k)$  where  $k$  is the element in row  $i$  and column  $j$ .

A *partial latin square* of order  $n$  is an  $n \times n$  array with entries chosen from a set  $N$  of size  $n$  such that each element of  $N$  occurs at most once in each row and column. We shall also use the corresponding set of triples to represent a partial latin square.

A partial latin square,  $P$ , of order  $n$  is *uniquely completable* (UC) if there is only one latin square,  $L$ , of order  $n$  that contains  $P$ .

The addition of a triple  $t = (i, j, k)$  to a partial latin square,  $P$ , is said to be *forced* if one of the following holds.

1.  $\forall h \neq i, \exists z$  such that  $(h, j, z)$  or  $(h, z, k) \in P$ .
2.  $\forall h \neq j, \exists z$  such that  $(z, h, k)$  or  $(i, h, z) \in P$ .
3.  $\forall h \neq k, \exists z$  such that  $(i, z, h)$  or  $(z, j, h) \in P$ .

A UC set,  $U$ , is *strong* if we can find a sequence of sets of triples  $U = S_1 \subset S_2 \subset \dots \subset S_r = L$  such that each triple  $t \in S_{v+1} - S_v$  is forced in  $S_v$ . A UC set that is not strong is *weak*. A UC set that contains no smaller UC set is called *critical*.

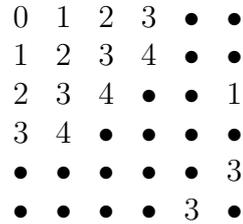
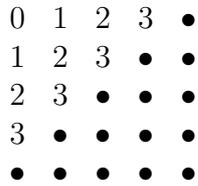


Figure 1: Strong critical set for  $C_5$       Figure 2: Weak UC set for  $C_6$

We will represent a cyclic latin square of order  $n$  by the set of triples

$$C_n = \{(i, j, i + j) \mid i = 0, 1, \dots, n - 1; j = 0, 1, \dots, n - 1\}$$

where addition is modulo  $n$ .

A family of strong UC sets of cyclic latin squares identified by Nelder [5] have been shown to be critical by Donovan and Cooper [2]; Figure 1 shows for  $C_5$ . (• represents an empty cell). In [4] Keedweel showed that weak UC sets do not exist in latin squares of order less than 5 . In [1] we reported that a computer search had found that no weak UC set exists in the cyclic latin square of order 5, and described weak UC sets for all cyclic latin squares of higher orders; Figure 2 shows the set for  $C_6$ . In this paper we identify critical subsets of these UC sets by extending the techniques used by Donovan and Cooper in [2].

## 2 Latin interchanges

The number of filled cells in a partial latin square defines its *size*; their positions define its *shape*. Two partial latin squares,  $P_1$  and  $P_2$ , of the same order, shape and size are *mutually balanced* if the entries in each row (and column) of  $P_1$  are the same as those in the corresponding row (and column) of  $P_2$ . They are *disjoint* if no cell of  $P_1$  has the same entry as the corresponding cell of  $P_2$ . If they are mutually balanced and disjoint they are called *disjoint mates*. A *latin interchange* is a partial latin square for which there exists a disjoint mate. Examples of a latin interchange and its disjoint mate are shown, superimposed in a single array, in Figure 3.

The motivation for studying latin interchanges arises from the fact that a partial latin square,  $P$ , is a UC set for a latin square,  $L$ , if and only if  $P$  intersects all latin interchanges contained in  $L$ .

An entry of a UC set for  $L$  is *crucial* if there is a latin interchange in  $L$  that intersects the UC set in that entry only; the latin interchange in Figure 3 intersects the critical set of  $C_5$  shown in Figure 1 only in the cell (0, 2). It is easy to see that a UC set is critical if and only if each of its entries is crucial.

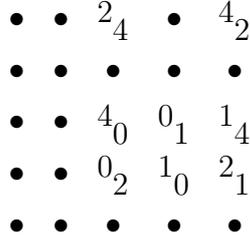


Figure 3: A latin interchange and its disjoint mate

The following lemma about latin interchanges in cyclic latin squares is from [2].

**Lemma 1** *If  $P_1$  is a latin interchange in  $C_n$ , then its transpose*

$$P_1^T = \{(j, i, k) \mid (i, j, k) \in P_1\}$$

*and, for any integers  $\alpha$  and  $\beta$ ,*

$$Q_1 = \{(i + \alpha, j + \beta, k + \alpha + \beta) \mid (i, j, k) \in P_1\}$$

*are also latin interchanges.*

Proof: If  $P_2$  is a disjoint mate of  $P_1$ , then  $P_2^T$  is a disjoint mate of  $P_1^T$ , and  $Q_2 = \{(i + \alpha, j + \beta, k + \alpha + \beta) \mid (i, j, k) \in P_2\}$  is a disjoint mate of  $Q_1$ .  $\square$ .

We note in passing that the above Lemma generalises easily to latin squares based on abelian groups.

### 3 Strong critical sets in cyclic latin squares

The following Lemma is well known.

**Lemma 2** *The set of triples*

$$S_n = \{(i, j, i + j) \mid i = 0, \dots, n - 2; j = 0, \dots, n - 2 - i\}$$

*is a strong UC set for  $C_n$ , for all  $n \geq 2$ .*

Proof: Consider the columns 0 to  $n - 1$  in order. The triples  $(n - 1 - i, j, j - i - 1)$  are forced as  $i$  ranges from  $j$  to 0.  $\square$

In the rest of this section we briefly describe Donovan and Cooper's proof that  $S_n$  is critical [2]. They showed that every entry in  $S_n$  is crucial by constructing a latin interchange in  $C_n$  that intersects  $S_n$  in only that entry. The bulk of the proof comprises several constructions that are used to show that each entry in row 0 of each  $S_n$  is crucial. Lemma 1 is then used to prove that the remaining entries are crucial.

**Theorem 1 (Donovan and Cooper)**  $S_n$  is a strong critical set for  $C_n$

Proof: We shall construct a latin interchange  $S_{r,c,n}$  that intersects  $S_n$  in only cell  $(r, c)$ . Figure 3 displayed  $S_{0,2,5}$  and its disjoint mate.

There are seven different constructions that we use to prove entries in row 0 and, for all  $c \in \{0, \dots, n-2\}$ , column  $c$  of each  $S_n$  are crucial. First we define  $x, u$  and  $v$ :

$$x = n - 1 - c; \quad x \equiv u \pmod{c+1}, 0 \leq u \leq c; \quad n \equiv v \pmod{x}, 0 \leq v < x.$$

We now choose a construction for finding each  $S_{0,c,n}$ .

1. If  $c = 0$ , use Construction 1.
2. If  $c = n - 2$ , use Construction 2.
3. If  $1 \leq c \leq n/2 - 1$  and  $u = 0$ , use Construction 3.
4. If  $n/2 \leq c \leq n - 3$  and  $v = 0$ , use Construction 4.
5. If  $1 \leq c \leq n/2 - 1$  and  $0 < u \leq (n - x)/2$ , use Construction 5.
6. If  $1 \leq c \leq n/2 - 1$  and  $u > (n - x)/2$ , use Construction 6.
7. If  $n/2 \leq c \leq n - 3$  and  $v \neq 0$ , use Construction 7.

### Construction 1

The partial latin square comprising all the entries of 0 and  $n - 1$  in  $C_n$  is a latin interchange intersecting  $S_n$  in only  $(0, 0)$ . Formally  $S_{0,0,n}$  is the set

$$\{(i, n - i, 0), (i, n - i - 1, n - 1) \mid i = 0, \dots, n - 1\}.$$

### Construction 2

A latin interchange intersecting  $S_n$  in only  $(0, n - 2)$  is formed by the set

$$\{(i, n - 2, i - 2), (i, n - 1, i - 1) \mid i = 0, \dots, n - 1\}.$$

This is the partial latin square comprising all entries in the last two columns of  $C_n$ .

### Construction 3

For  $1 \leq c \leq n/2 - 1$  and  $u = 0$ , a latin interchange exists which intersects  $S_n$  in only  $(0, c)$  with cells containing either  $c$  or  $n - 1$ .  $S_{0,c,n}$  is the set

$$\{(i(c+1), n-1-(i-1)(c+1), c), (i(c+1), n-1-i(c+1), n-1) \mid i = 0, \dots, x/(c+1)\}.$$

#### Construction 4

For  $n/2 \leq c \leq n - 3$  and  $v = 0$ , a latin interchange intersecting  $S_n$  in only  $(0, c)$  and containing entries in only columns  $c$  and  $n - 1$  can be found.  $S_{0,c,n}$  is the set

$$\{(ix, c, ix + c), (ix, n - 1, ix - 1) \mid i = 0, \dots, (n/x) - 1\}.$$

#### Construction 5

For  $1 \leq c \leq n/2 - 1$  and  $0 < u \leq (n - x)/2$  we construct a partial latin square similar to that in Construction 3, but because  $u \neq 0$ , we “add” the latin interchange:  $S_{0,c-u,c+1}$ . Note that  $0 \leq c - u \leq c - 1$ , and  $c + 1 < n$  so we can choose one of the 7 constructions to obtain  $S_{0,c-u,c+1}$ .

Define  $p = \lfloor x/(c + 1) \rfloor$  and

$$R = \{(i + u, j + x, i + j + u + x) \mid (i, j, k) \in S_{0,c-u,c+1}\}.$$

$S_{0,c,n}$  is the set

$$\begin{aligned} &\{(0, c, c), (0, n - 1, n - 1)\} \cup R \cup \\ &\{(u + m(c + 1), n - 1 - u - m(c + 1), n - 1), \\ &\quad (u + m(c + 1), n - 1 - u - (m - 1)(c + 1), c) \mid m = 1, \dots, p\}. \end{aligned}$$

#### Construction 6

For  $1 \leq c \leq n/2 - 1$  and  $u > (n - x)/2$  we use a method similar to that employed in the previous construction. We require the latin interchange  $S_{0,u-1,c+1}$ , note that this can be obtained using one of the 7 constructions and  $c + 1 < n$ . Define  $p = \lfloor x/(c + 1) \rfloor$  and

$$R = \{i + 1, j - 1 - u, i + j - u \mid (i, j, k) \in S_{0,u-1,c+1}^T\}.$$

$S_{0,c,n}$  is the set

$$\begin{aligned} &\{(0, c, c), (0, n - 1, n - 1)\} \cup R \cup \\ &\{(u + m(c + 1), n - 1 - u - (m - 1)(c + 1), c), \\ &\quad (u + m(c + 1), n - 1 - u - m(c + 1), n - 1) \mid m = 1, \dots, p\}. \end{aligned}$$

#### Construction 7

For  $n/2 \leq c \leq n - 3$  and  $v \neq 0$  use a latin interchange  $S_{0,x-1,x+v}$ . Note that this can be obtained using one of the 7 constructions and  $x + v < n$ . Define  $q = \lfloor (n - x)/x \rfloor$  and

$$R = \{((q - 1)x + i + 1, c + j, (q - 2)x + i + j) \mid (i, j, k) \in S_{x-1,x+v}^T\}.$$

$S_{c,n}$  is the set

$$R \cup \{(mx, c, mx + c), (mx, n - 1, mx - 1) \mid m = 0, \dots, q - 1\}.$$

In [2], each of these constructions was shown to be a latin interchange by the construction of a disjoint mate. As the latin interchanges required for the last three constructions are of order less than  $n$ , we can prove by induction that we can construct  $S_{0,c,n}$  for all  $c$ , for all  $n$ . It was also shown that all the latin interchanges produced with these constructions have no entries in columns 0 to  $c - 1$ ; and, if  $c \leq n/2 - 1$ , all entries are in rows 0 to  $x$ ; if  $c > n/2 - 1$ , all entries are in rows 0 to  $c + 1$ .

We complete the proof using Lemma 1. If  $c \leq n/2 - 1$  and  $1 \leq s \leq c$ , or if  $c > n/2 - 1$  and  $1 \leq s \leq n - c - 2$ , then

$$S_{s,c,n} = \{(i + s, j, k + s) \mid (i, j, k) \in S_{0,c,n}\}$$

The remaining entries of  $S_n$  are all in cells  $(r, c)$  such that the entry in  $(c, r)$  has been shown to be crucial. Therefore we can define

$$S_{r,c,n} = \{(j, i, k) \mid (i, j, k) \in S_{c,r,n}\}$$

As  $S_n^T = S_n$  this will be a latin interchange that intersects  $S_n$  in only cell  $(r, c)$ .  
□

## 4 Weak critical sets in cyclic latin squares

**Lemma 3** *The set of triples  $W_n = P_n \cup Q_n \cup R_n$ , where*

$$\begin{aligned} P_n &= \{(i, j, i + j) \mid i = 0, \dots, n - 4; j = 0, \dots, n - 4 - i\} \\ Q_n &= \{(i, n - 2 - i, n - 2) \mid i = 1, \dots, n - 3\} \\ R_n &= \{(2, n - 1, 1), (n - 2, n - 1, n - 3), (n - 1, n - 2, n - 3)\} \end{aligned}$$

*is a weak UC set for  $C_n$ .*

Proof: The triples  $(i, n - 3 - i, n - 3), i = 0, \dots, n - 3$  are forced; the resulting set is the weak UC set for  $C_n$  obtained in [1]. □

$W_n$  is a subset of the weak UC set introduced in [1]; compare Figures 2 and 4. The principal result of this paper follows.

**Theorem 2**  *$W_n$  is a critical set.*

The proof of this theorem uses the constructions of the previous section where possible. Some further constructions are required. We define  $W_{r,c,n}$  as a latin interchange in  $C_n$  that intersects  $W_n$  only in the cell  $(r, c)$ .

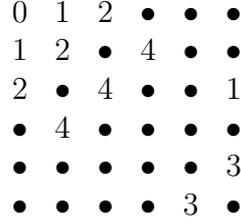


Figure 4:  $W_6$

We first consider entries in row 0 of  $P_n$ . If  $S_{0,c,n}$  intersects  $R_n$  we cannot let  $W_{r,c,n} = S_{r,c,n}$ . However,  $S_{0,c,n}$  will only intersect  $\{(n-2, n-1), (n-1, n-2)\}$  if  $c = n-2$  or  $c = n-1$ , and neither of these entries are in  $W_n$ . Therefore we need only consider whether  $S_{r,c,n}$  intersects  $(2, n-1)$ . Constructions 8–12 deal with all such cases.

For squares of all orders,  $S_{0,1,n}$  intersects  $W_n$  in the cell  $(2, n-1, 1)$ , so we must define a distinct  $W_{0,1,n}$ . For even  $n$  we use Construction 8; for odd  $n$  Construction 9.

### Construction 8

$$\begin{aligned}
W_{0,1,n} = & \{(n+1-2i, 2i, 1), (n-1-2i, 2i, n-1) \mid i = 1, \dots, n/2-2\} \cup \\
& \{(n-2, i+2, i), (n-1, i+2, i+1), (i+2, n-2, i), \\
& \quad (i+2, n-1, i+1) \mid i = 1, \dots, n-5\} \cup \\
& \{(0, 1, 1), (0, n-2, n-2), (0, n-1, n-1), (n-2, 0, n-2), \\
& \quad (n-2, 1, n-1), (n-2, n-2, n-4), \\
& \quad (n-1, 0, n-1), (n-1, n-1, n-2)\}
\end{aligned}$$

and has disjoint mate

$$\begin{aligned}
& \{(n+1-2i, 2i, n-1), (n-1-2i, 2i, 1) \mid i = 1, \dots, n/2-2\} \cup \\
& \{(n-2, i+2, i+1), (n-1, i+2, i), (i+2, n-2, i+1), \\
& \quad (i+2, n-1, i) \mid i = 2, \dots, n-5\} \cup \\
& \{(0, 1, n-1), (0, n-2, 1), (0, n-1, n-2), (3, n-2, 2), (3, n-1, n-1), \\
& \quad (n-2, 0, n-1), (n-2, 1, 1), (n-2, 3, 2), (n-2, n-2, n-2), \\
& \quad (n-1, 0, n-2), (n-1, 3, 1), (n-1, n-1, n-4)\}.
\end{aligned}$$

**Example 1**  $W_{0,1,12}$  and its disjoint mate are displayed in Figure 6.

### Construction 9

$$\begin{aligned}
W_{0,1,n} = & \{(2i+1, n-2-2i, n-1), (2i+1, n-2i, 1) \mid i = 1, \dots, (n-3)/2\} \cup \\
& \{(n-2, i+2, i), (n-1, i+1, i), (i+2, n-2, i), \\
& \quad (i+1, n-1, i) \mid i = 2, \dots, n-4\} \cup \\
& \{(0, 1, 1), (0, n-2, n-2), (0, n-1, n-1), (n-2, 0, n-2), \\
& \quad (n-2, 1, n-1), (n-1, 0, n-1), (n-1, n-1, n-2)\}
\end{aligned}$$

and has disjoint mate

$$\begin{aligned} & \{(2i+1, n-2-2i, 1), (2i+1, n-2i, n-1) \mid i=2, \dots, (n-5)/2\} \cup \\ & \{(n-2, i+1, i), (n-1, i+2, i), (i+2, n-1, i), \\ & \quad (i+1, n-2, i) \mid i=2, \dots, n-5\} \cup \\ & \{(0, 1, n-1), (0, n-2, 1), (0, n-1, n-2), (3, n-4, 1), (3, n-1, n-1), \\ & \quad (n-3, n-2, n-4), (n-2, 0, n-1), (n-2, 1, 1), (n-2, n-3, n-4), \\ & \quad (n-2, n-2, n-2), (n-1, 0, n-2), (n-1, 3, n-1), (n-1, n-1, n-4)\}. \end{aligned}$$

**Example 2**  $W_{0,1,11}$  and its disjoint mate are displayed in Figure 7.

When Construction 5 is used to obtain  $S_{0,c,n}$  and  $u=1$  we cannot let  $W_{0,c,n} = S_{0,c,n}$  as  $S_{0,c,n}$  includes an entry in cell  $(2, n-1, 1)$ . Instead we use the following construction.

**Construction 10**

Let  $p = \lfloor x/(n-x) \rfloor$  and then

$$\begin{aligned} W_{0,c,n} = & \{(0, c, c), (0, n-2, n-2), (0, n-1, n-1), (n-1, n-1, n-2)\} \cup \\ & \{(1+m(c+1), c(p+1-m) + (p-m), n-1), \\ & \quad (1+m(c+1), c(p+2-m) + (p+1-m), c) \mid m=1, \dots, p\} \cup \\ & \{(i, n-1, i-1), (i+1, n-2, i-1) \mid i=c+2, \dots, n-3\} \cup \\ & \{(n-2, i, i-2), (n-1, i, i-1) \mid i=0, \dots, n-3\}. \end{aligned}$$

This has the disjoint mate

$$\begin{aligned} & \{(0, c, n-1), (0, n-2, c), (0, n-1, n-2)\} \cup \\ & \{(1+m(c+1), c(p+1-m) + (p-m), c), \\ & \quad (1+m(c+1), c(p+2-m) + (p+1-m), n-1) \mid m=2, \dots, p\} \cup \\ & \{(i, n-2, i-1), (i+1, n-1, i-1) \mid i=c+2, \dots, n-4\} \cup \\ & \{(n-2, i, i-1), (n-1, i, i-2) \mid i=0, \dots, n-3\} \cup \\ & \{(c+2, p(c+1)-1, c), (c+2, n-1, n-1), (n-2, n-2, n-2), \\ & \quad (n-3, n-2, n-4), (n-1, n-1, n-4)\}. \end{aligned}$$

**Example 3**  $W_{0,3,13}$  and its disjoint mate are displayed in Figure 8.

If  $u=2$  we cannot use Constructions 5 or 6 as they intersect the cell  $(2, n-1)$ . The alternative construction used depends on whether  $c$  is odd or even. For odd  $c$ , we use Construction 11; for even  $c$  we use Construction 12.

**Construction 11**

Again, let  $p = \lfloor x/(n-x) \rfloor$ ,

$$\begin{aligned} W_{0,c,n} = & \{(2+m(c+1), n-3-(m-1)(c+1), c), \\ & \quad (2+m(c+1), n-3-m(c+1), n-1) \mid m=1, \dots, p\} \cup \\ & \{(i, n-1-i, n-1), (i+1, n-1-i, 0) \mid i=0, 1, 2\} \cup \\ & \{(2i+1, n-1, 2i), (2i+3, n-3, 2i) \mid i=1, \dots, (c-1)/2\} \cup \\ & \{(0, c, c), (c+1, n-1, c), (c+1, n-2, c-1), (c+2, n-2, c)\} \end{aligned}$$

and has disjoint mate

$$\begin{aligned}
W_{0,c,n} = & \{(2 + m(c + 1), n - 3 - (m - 1)(c + 1), n - 1), \\
& (2 + m(c + 1), n - 3 - m(c + 1), c) \mid m = 1, \dots, p\} \cup \\
& \{(i, n - 1 - i, 0), (i, n - i, n - 1), (c + i, n - 1 - i, c), \\
& (c + i, n - i, c - 1) \mid i = 1, 2\} \cup \\
& \{(2i + 1, n - 3, 2i), (2i + 1, n - 1, 2i - 2) \mid i = 1, \dots, (c - 1)/2\} \cup \\
& \{(0, c, n - 1), (0, n - 1, c)\}.
\end{aligned}$$

**Example 4**  $W_{0,3,14}$  and its disjoint mate are displayed in Figure 9.

### Construction 12

Let  $p = \lfloor x/(n - x) \rfloor$ ,

$$\begin{aligned}
W_{0,c,n} = & \{(2 + m(c + 1), n - 3 - (m - 1)(c + 1), c), \\
& (2 + m(c + 1), n - 3 - m(c + 1), n - 1) \mid m = 1, \dots, p\} \cup \\
& \{(i, n - 1 - i, n - 1), (i + 1, n - 1 - i, 0) \mid i = 0, 1, 2\} \cup \\
& \{(2i + 1, n - 1, 2i), (2i + 3, n - 3, 2i) \mid i = 1, \dots, (c - 2)/2\} \cup \\
& \{(0, c, c), (c + 1, n - 1, c)\}
\end{aligned}$$

and has disjoint mate

$$\begin{aligned}
W_{0,c,n} = & \{(2 + m(c + 1), n - 3 - (m - 1)(c + 1), n - 1), \\
& (2 + m(c + 1), n - 3 - m(c + 1), c) \mid m = 1, \dots, p\} \cup \\
& \{(i, n - 1 - i, 0), (i, n - i, n - 1) \mid i = 1, 2\} \cup \\
& \{(2i + 1, n - 3, 2i), (2i + 1, n - 1, 2i - 2) \mid i = 1, \dots, c/2\} \cup \\
& \{(0, c, n - 1), (0, n - 1, c)\}.
\end{aligned}$$

**Example 5**  $W_{0,6,16}$  and its disjoint mate are displayed in Figure 10.

We now look at the entries of  $Q_n$ . For such entries  $S_{r,c,n}$  is constructed using Construction 4 or 7, and “translating” it using Lemma 1.  $S_{2,n-4,n}$  will always intersect  $(2, n - 1)$ ; Constructions 13, 14 and 15 present an alternative. For the other entries of  $R_n$  there is a possibility that  $S_{r,c,n}$  will intersect  $\{(n - 2, n - 1), (n - 1, n - 2)\}$ . This possibility is removed by adapting Construction 7 (Construction 4 will never intersect these two cells). To construct a latin interchange using Construction 7 a smaller construction is required, and that itself may have been based on a smaller construction and so on. Only if the “basic” construction was an instance of Construction 2 will  $S_{r,c,n}$  intersect those two cells. Therefore we replace Construction 2 with the set

$$\begin{aligned}
& \{(i, n - 2, i - 2), (i, n - 1, i - 1), (n - 2, i, i - 2), \\
& (n - 1, i, i - 1) \mid i = 0, \dots, n - 3\} \cup \\
& \{(n - 2, n - 2, n - 4), (n - 1, n - 1, n - 2)\}
\end{aligned}$$

The entry in cell  $(2, n - 4)$  is a particularly awkward case requiring three constructions. If  $n \equiv 0 \pmod{3}$  we use Construction 13; if  $n \equiv 1 \pmod{3}$  we use Construction 14; if  $n \equiv 2 \pmod{3}$  we use Construction 15.

**Construction 13**

$$\begin{aligned}
W_{2,n-4,n} = & \{(0, n-2, n-2), (0, n-1, n-1), (1, n-2, n-1), \\
& (1, n-1, 0), (2, n-4, n-2), (2, n-2, 0), (3, n-3, 0), \\
& (3, n-1, 2), (4, n-4, 0), (4, n-3, 1), (5, n-4, 1), \\
& (5, n-3, 2), (5, n-1, 4)\} \cup \\
& \{(5+3i, n-4, 1+3i), (5+3i, n-1, 4+3i) \mid i = 0, \dots, n-6/3\}.
\end{aligned}$$

It has the disjoint mate

$$\begin{aligned}
& \{(0, n-2, n-1), (0, n-1, n-2), (1, n-2, 0), \\
& (1, n-1, n-1), (2, n-4, 0), (2, n-2, n-2), (3, n-3, 2), \\
& (3, n-1, 0), (4, n-4, 1), (4, n-3, 0), (5, n-4, 4), \\
& (5, n-3, 1), (5, n-1, 2)\} \cup \\
& \{(5+3i, n-4, 4+3i), (5+3i, n-1, 1+3i) \mid i = 1, \dots, n-6/3\}.
\end{aligned}$$

**Example 6**  $W_{2,8,12}$  is displayed in Figure 11.

**Construction 14**

$$\begin{aligned}
W_{2,n-4,n} = & \{(0, n-2, n-2), (0, n-1, n-1), (1, n-2, n-1), \\
& (1, n-1, 0), (2, n-4, n-2), (2, n-2, 0), (3, n-3, 0), \\
& (3, n-2, 1), (3, n-1, 2), (4, n-4, 0), (4, n-3, 1), \\
& (4, n-2, 2)\} \cup \\
& \{(6+3i, n-4, 2+3i), (6+3i, n-1, 5+3i) \mid i = 0, \dots, n-7/3\}.
\end{aligned}$$

It has the disjoint mate

$$\begin{aligned}
& \{(0, n-2, n-1), (0, n-1, n-2), (1, n-2, 0), \\
& (1, n-1, n-1), (2, n-4, 0), (2, n-2, n-2), (3, n-3, 1), \\
& (3, n-2, 2), (3, n-1, 0), (4, n-4, 2), (4, n-3, 0), (4, n-2, 1), \} \cup \\
& \{(6+3i, n-4, 5+3i), (6+3i, n-1, 2+3i) \mid i = 0, \dots, n-7/3\}.
\end{aligned}$$

**Example 7**  $W_{2,9,13}$  is displayed in Figure 12.

**Construction 15**

$$\begin{aligned}
W_{2,n-4,n} = & \{(0, n-2, n-2), (0, n-1, n-1), (1, n-2, n-1), \\
& (1, n-1, 0), (2, n-4, n-2), (2, n-2, 0)\} \cup \\
& \{(4+3i, n-4, 3i), (4+3i, n-1, 3(i+1)) \mid i = 0, \dots, n-5/3\}.
\end{aligned}$$

It has the disjoint mate

$$\begin{aligned}
& \{(0, n-2, n-1), (0, n-1, n-2), (1, n-2, 0), \\
& (1, n-1, n-1), (2, n-4, 0), (2, n-2, n-2)\} \cup \\
& \{(4+3i, n-4, 3(i+1)), (4+3i, n-1, 3i) \mid i = 0, \dots, n-5/3\}.
\end{aligned}$$

**Example 8**  $W_{2,10,14}$  is displayed in Figure 13.

The following constructions deal with the entries in  $R_n$

**Construction 16**

$W_{2,n-1,n}$  is the set

$$\{(i, n-2, i-2), (i, n-1, i-1), (n-2, i, i-2), \\ (n-1, i, i-1) \mid i = 0, \dots, n-3\} \cup \\ \{(n-2, n-2, n-4), (n-1, n-1, n-2)\}.$$

It has the disjoint mate

$$\{(i, n-2, i-1), (i, n-1, i-2), (n-2, i, i-1), \\ (n-1, i, i-2) \mid i = 0, \dots, n-3\} \cup \\ \{(n-2, n-2, n-2), (n-1, n-1, n-4)\}.$$

**Example 9**  $W_{2,6,7}$  displayed in Figure 14.

The latin interchange that we use for the remaining two cells depends on whether  $n$  is even or odd. For even orders we use Construction 17; for odd orders Construction 18.

**Construction 17**

$W_{n-2,n-1,n}$  is the intercalate

$$\{(n/2-2, n/2-1, n-3), (n/2-2, n-1, n/2-3), \\ (n-2, n/2-1, n/2-3), (n-2, n-1, n-3)\}.$$

Its transpose is  $W_{n-1,n-2,n}$

**Example 10** Figure 15 displays  $W_{8,9,10}$  and  $W_{9,8,10}$ .

We can not use this construction for the cell  $(6, 7, 5)$  in  $C_8$  because the intercalate includes  $(2, 7, 1)$  which is in  $W_8$ . We display the latin interchange that we use instead in Figure 5.

**Construction 18**

$W_{n-1,n-2,n}$  is the set

$$\{(2i, n-3-2i, n-3), (2i, n-1-2i, n-1) \mid i = 0, \dots, (n-1)/2\} \cup \\ \{(0, n-2, n-2), (n-1, n-1, n-2)\}$$

It has disjoint mate

$$\{(2i, n-3-2i, n-1), (2i, n-1-2i, n-3) \mid i = 1, \dots, (n-3)/2\} \cup \\ \{(0, n-3, n-1), (0, n-2, n-3), (0, n-1, n-2), \\ (n-1, 0, n-3), (n-1, n-2, n-2), (n-1, n-1, n-1)\}.$$

Its transpose is  $W_{n-2,n-1,n}$ .

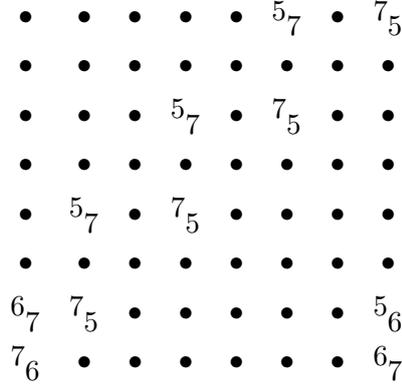


Figure 5:  $W_{6,7,8}$

**Example 11** In Figure 16 we display  $W_{8,7,9}$ .

We can now prove Theorem 2. We have given constructions that show for each  $n$ , the entries of  $Q_n$  and  $R_n$ , and the top row entries of  $P_n$  are crucial. We now deal with all entries of  $P_n$  outside row 0. For entries in column 0, including that in cell  $(0,0)$ , we let  $W_{r,0,n} = S_{r,0,n}$ . For entries in the set

$$\{(r, c) \mid r = 1, \dots, n/2 - 2; c = i, \dots, n - 4 - i\}$$

we define  $W_{r,c,n}$  as the set

$$\{(i + r, j - 1, k + r - 1) \mid (i, j, k) \in S_{0,c+1,n}\}$$

which, by Lemma 1 is a latin interchange in  $C_n$ . It can easily be seen that it does not intersect  $P_n$  or  $Q_n$  except in cell  $(r, c)$ . It can also be deduced that its entries are in columns  $c$  to  $n - 2$ . If  $c \leq n/2$  entries are in rows  $r$  to  $n - c - 1$ , and as  $c$  is at least 1, there are no entries in row  $n - 1$ . If  $c > n/2$ , entries are in rows 0 to  $r + c + 1$ . As  $r + c$  is at most  $n - 4$ , there are again no entries in row  $n - 1$ . Therefore this set does not intersect  $R_n$ .

For the remainder of the entries in  $P_n$  outside row 0,  $W_{r,c,n}$  is the transpose of  $W_{c,r,n}$ .  $\square$

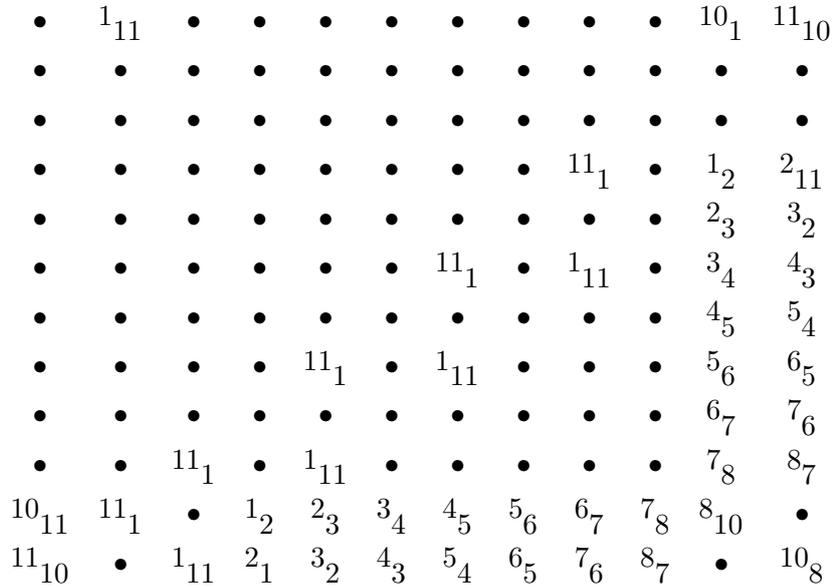


Figure 6:  $W_{0,1,12}$

## References

- [1] D. Bedford and M. Johnson, Weak uniquely completable sets for finite groups, Bulletin of the London Math. Soc., to appear
- [2] D. Donovan and J. Cooper, Critical sets in back circulant latin squares, Aequationes Mathematicae **52** (1996), pp.157-179.
- [3] A. D. Keedwell, Critical sets and critical partial latin squares, in Combinatorics, Graph Theory, Algorithms and Applications, World Scientific Publ. Co (1993), pp.111-123.
- [4] A. D. Keedwell, What is the size of the smallest latin square for which a weakly completable critical set of cells exists?, Ars Combinatoria, to appear.
- [5] J. Nelder, Critical Sets in latin squares, CSIRO Division of Math. and Stats., Newsletter **38** (1977).

## A Examples

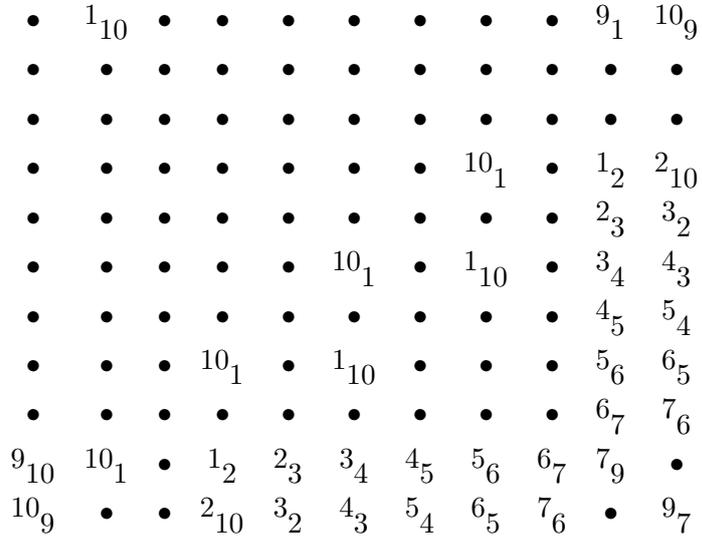


Figure 7:  $W_{1,11}$

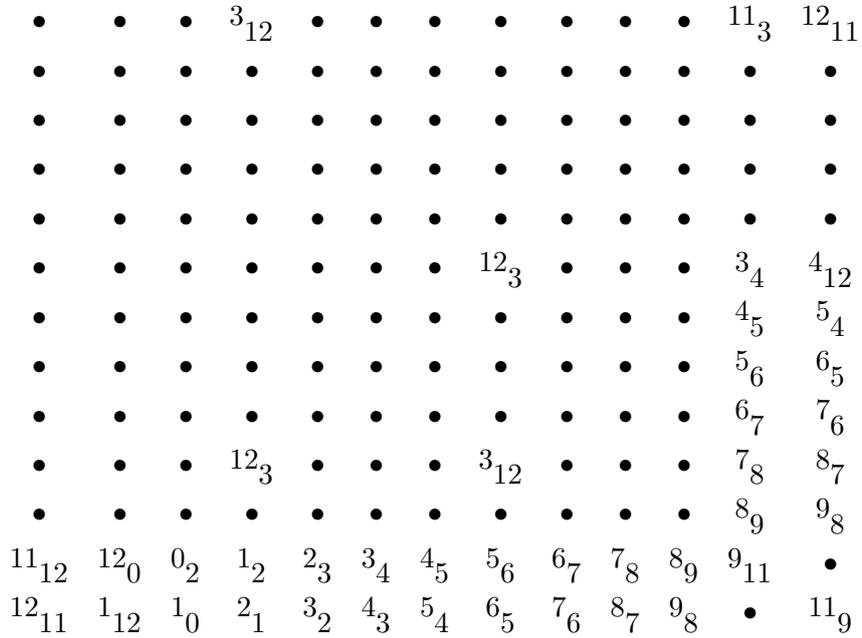


Figure 8:  $W_{0,3,13}$

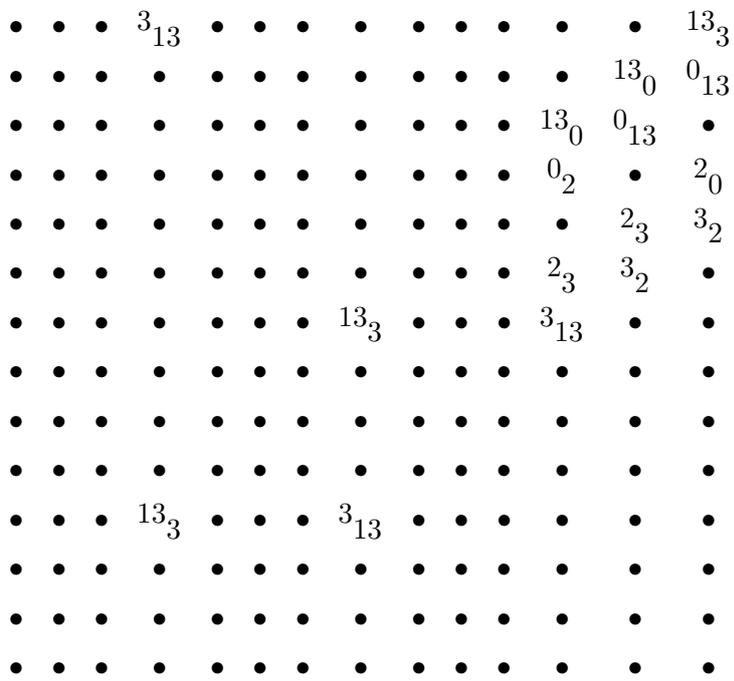


Figure 9:  $W_{0,3,14}$

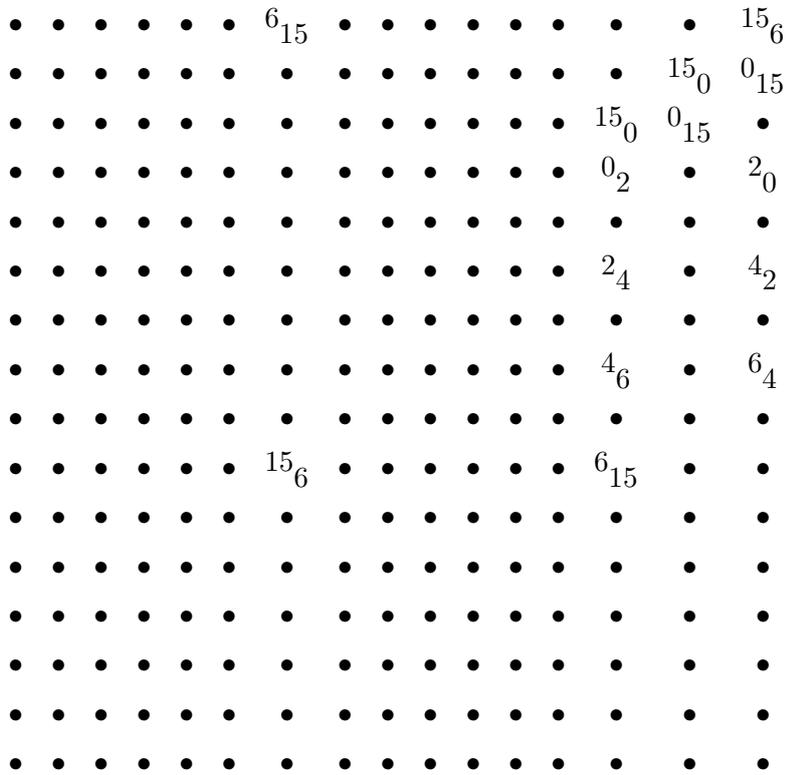


Figure 10:  $W_{0,6,16}$

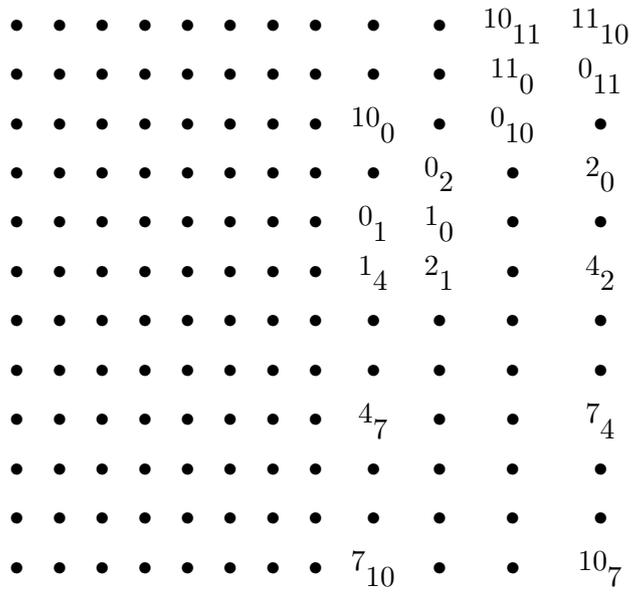


Figure 11:  $W_{2,8,12}$

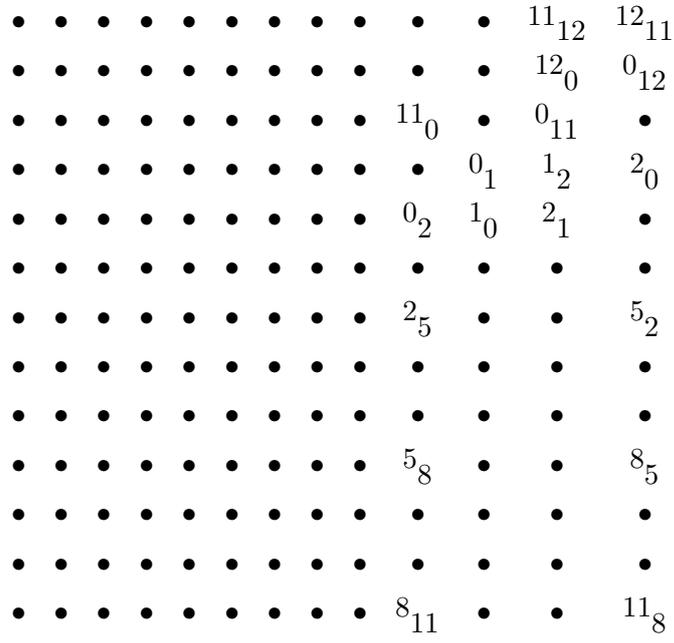


Figure 12:  $W_{2,9,13}$

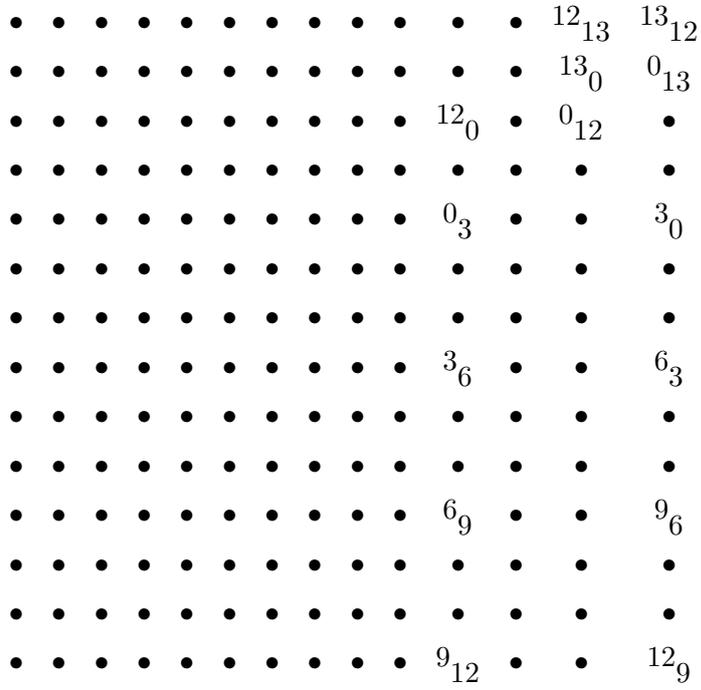


Figure 13:  $W_{2,10,14}$

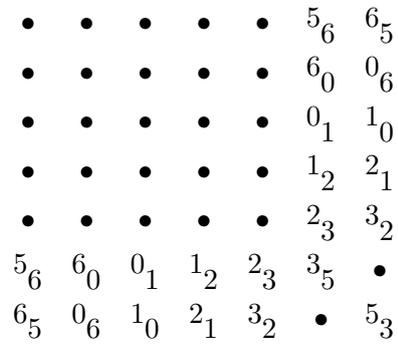


Figure 14:  $W_{2,6,7}$

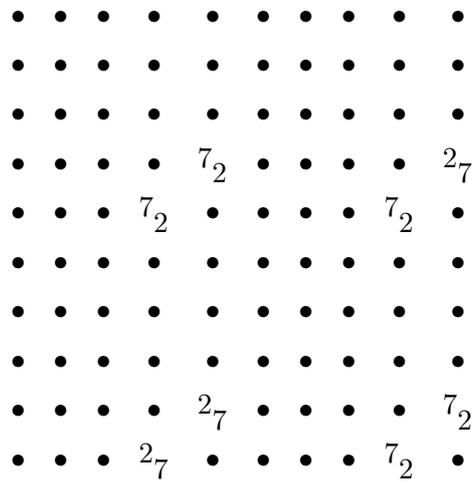


Figure 15:  $W_{8,9,10}$  and  $W_{9,8,10}$

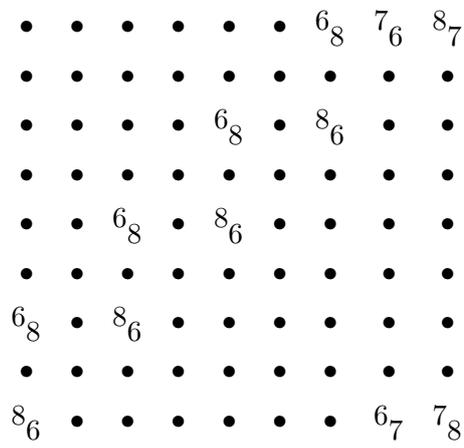


Figure 16:  $W_{8,7,9}$