Cycle decompositions of the complete graph

A.J.W. Hilton
Department of Mathematics
University of Reading
Whiteknights
P.O. Box 220
Reading RG6 6AX
U.K.

Matthew Johnson
Department of Mathematics
London School of Economics
Houghton Street
London WC2A 2AE
U.K.
e-mail: matthew@maths.lse.ac.uk

Abstract

For a positive integer n, let G be K_n if n is odd and K_n less a one-factor if n is even. In this paper it is shown that, for non-negative integers p, q and r, there is a decomposition of G into p 4-cycles, q 6-cycles and r 8-cycles if $4p + 6q + 8r = |E(G)|$, $q = 0$ if $n < 6$ and $r = 0$ if $n < 8$.

1 Introduction

Is it possible to decompose K_n (n odd) or $K_n - I_n$ (n even, I_n is a one-factor of K_n) into t cycles of lengths $m_1, \ldots, m_t$? Obvious necessary conditions for finding these *cycle decompositions* are that each cycle length must be between 3 and n and the sum of the cycle lengths must equal the number of edges in the graph being decomposed. That these simple conditions are sufficient was conjectured by Alspach [3] in 1981. To date, only a few special cases have been solved, mostly where each m_i must take one of a restricted number of values [1, 2, 6, 7]. In particular, we note that the case where all the cycles have the same length has recently been completely solved by Alspach and Gavlas [4] and Šajna [10]. We also note that Rosa [9] has proved that the conjecture is true for $n \leq 10$, and Balister [5] has shown that the conjecture is true if the cycle lengths are bounded by some linear function of n and n is sufficiently large.

In this paper, we solve the case where each cycle has length 4, 6 or 8; for the proof we introduce an innovative extension technique for finding cycle decompositions of $K_n(-I_n)$ from decompositions of K_m, $m < n$.

Theorem 1 *Let n be a positive integer. Let p, q and r be non-negative integers. Then K_n (n odd) or $K_n - I_n$ (n even) can be decomposed into p 4-cycles, q 6-cycles and r 8-cycles if and only if*

1. $4p + 6q + 8r = \begin{cases} |E(K_n)| \text{ if } n \text{ is odd,} \\ |E(K_n - I_n)| \text{ if } n \text{ is even, and} \end{cases}$

2. *the cycles all have length at most n.*

Our novel extension technique is described in the next section. In Section 3 we consider cycle decompositions of $K_{m,n}$ as these will also be required in the proof of Theorem 1. This proof is in the final section.

Definitions and notation. An edge joining u and v is denoted (u, v). A path of length $k-1$ is denoted $(v_1, \ldots, v_k)$ where v_i is adjacent to v_{i+1}, $1 \le i \le k-1$, but a path of length zero—that is, a single vertex—will be denoted simply v_1 rather than (v_1). A k-cycle is denoted $[v_1, \ldots, v_k]$, where v_i is adjacent to v_{i+1}, $1 \le i \le k-1$, and v_0 is adjacent to v_k. A *path-graph* is a collection of disjoint paths and is described by listing the paths. A path-graph containing only paths of lengths zero or one is a *matching*.

2 An extension technique

In this section we introduce a technique that we can use to obtain cycle decompositions of $K_n(-I_n)$ from cycle decompositions of $K_m(-I_m)$ when $m < n$.

First we define a different type of decomposition. Let n, s and t be non-negative integers. An (s,t)-*decomposition* of K_n can be either even or odd. It contains the following collection of subgraphs:

- path-graphs $P_1, \ldots, P_s$,
- cycles $C_{s+1}, \ldots, C_{s+t}$, and
- if it is an even decomposition, a matching P_0;

with the following properties:

- their edge-sets partition the edge-set of K_n,
- each vertex of K_n is in precisely s of the subgraphs $P_1, \ldots P_s, C_{s+1}, \ldots, C_{s+t}$, and
- if it is an even decomposition, each vertex is in P_0.

Example 1. We display an odd $(4,2)$-decomposition of K_7:

$$\begin{aligned} P_1 &= (1,5,2,4),(3,7) \\ P_2 &= (1,6,2,7) \\ P_3 &= (3,6,5),2 \\ P_4 &= (4,7,5),6 \\ C_5 &= [1,3,5,4,6,7] \\ C_6 &= [1,2,3,4]. \end{aligned}$$

Theorem 2 *Let m, n, s and t be non-negative integers with $m < n$ and $s = \lfloor (n-1)/2 \rfloor$. Let $D = (P_0,)P_1, \dots P_s, C_{s+1}, \dots, C_{s+t}$ be an (s,t)-decomposition of K_m that is even or odd as the parity of n.*

If the vertices of K_m are identified with m of the vertices of K_n, then we can find a decomposition of K_n (less a one-factor I_n if n is even) into cycles $C_1, \dots, C_{s+t}$ where, for $1 \le i \le s$, C_i is a supergraph of P_i, and if n is even I_n is a supergraph of P_0, if and only if,

$$\begin{aligned} \text{for } 1 \le i \le s, \quad n-m &\ge |V(P_i)| - |E(P_i)|, \text{ and,} && (1)\\ \text{if } n \text{ is even}, \quad |E(P_0)| &\ge m - n/2. && (2) \end{aligned}$$

Notice that since each vertex of $V(K_n \setminus K_m)$ must be in s of the cycles, it must be in each C_i, $1 \le i \le s$, since the other cycles are subgraphs of K_m. Therefore C_i, $1 \le i \le s$, has length $|V(P_i)| + n - m$.

Before we prove Theorem 2, let us see how it can be used. We consider four examples.

Example 2. Let D be the $(4,2)$-decomposition of K_7 shown in Example 1. Apply Theorem 2 with $n = 9$, $m = 7$, $s = 4$ and $t = 2$. Checking that (1) is satisfied is easy if we notice that $|V(P_i)| - |E(P_i)|$ is equal to the number of paths in P_i (remember that we count an isolated vertex as a path). By Theorem 2, there exists a cycle decomposition $C_1, \dots, C_6$ of K_9 where C_i, $1 \le i \le 4$ is a supergraph of P_i. As C_i has length $|V(P_i)| + n - m$, C_1 will be an 8-cycle and C_2, C_3 and C_4 will be 6-cycles. We display an example of a cycle decomposition obtained by extending D.

$$\begin{aligned} C_1 &= [1,5,2,4,9,7,3,8]\\ C_2 &= [1,6,2,7,8,9]\\ C_3 &= [2,9,3,6,5,8]\\ C_4 &= [4,7,5,9,6,8]\\ C_5 &= [1,3,5,4,6,7]\\ C_6 &= [1,2,3,4]. \end{aligned}$$

In the following three examples, we begin with a cycle decomposition of $K_m(-I_m)$. By making slight changes to this decomposition—we take the edges from one of the cycles, or from the one-factor I_m, and use them to create path-graphs—we obtain an (s,t)-decomposition of K_m. Then we apply Theorem 2 to obtain a cycle decomposition of K_n for some $n > m$. This method of obtaining a cycle decomposition of a complete graph from a cycle decomposition of a smaller complete graph will help us to give an inductive proof of Theorem 1 in the final section.

Example 3. Let Δ be a decomposition of K_{10} into p 4-cycles, q 6-cycles and r 8-cycles and a one-factor I_{10} where the vertices are labelled so that

$$I_{10} = (1,2),(3,4),(5,6),(7,8),(9,10).$$

Label the cycles $C_7 \ldots, C_{6+p+q+r}$ and let

$$\begin{aligned} P_1 &= (1,2),(3,4),5 \\ P_2 &= (5,6),7 \\ P_3 &= (7,8),9 \\ P_4 &= (9,10),1 \\ P_5 &= 2,3,4 \\ P_6 &= 6,8,10. \end{aligned}$$

Let $D = P_1, \ldots, P_6, C_7, \ldots, C_{p+q+r}$ and notice that it is a decomposition of K_{10}. As the cycles $C_7, \ldots, C_{6+p+q+r}$ form a decomposition of $K_{10} - I_{10}$, each vertex $v \in V(K_{10})$ will be in four of them (consider degrees). Each vertex is also in two of the path-graphs displayed above. Thus each vertex is in 6 of the graphs of D, and D is an odd $(6, p+q+r)$-decomposition of K_{10}. Apply Theorem 2 with $n = 13$, $m = 10$, $s = 6$ and $t = p+q+r$ (it is easy to check that (1) is satisfied). The decomposition of K_{13} obtained contains all the cycles of D and also cycles $C_1, \ldots, C_6$ that are supergraphs of the path-graphs $P_1, \ldots, P_6$. Thus C_1 has length 8 and C_i, $2 \le i \le 6$, has length 6, and the decomposition of K_{13} contains p 4-cycles, $q+5$ 6-cycles and $r+1$ 8-cycles.

Hence, if we require a decomposition of K_{13} into p' 4-cycles, q' 6-cycles and r' 8-cycles, we can obtain it from a decomposition of K_{10} into $p = p'$ 4-cycles, $q = q' - 5$ 6-cycles and $r = r' - 1$ 8-cycles. Of course, we require that $q' \ge 5$ and $r' \ge 1$ so that p, q and r are non-negative.

Example 4. Let $m = 1 \bmod 4$, $m \ge 9$. Suppose that we have a decomposition Δ of K_m into p 4-cycles, q 6-cycles and r 8-cycles. We are going to use this to find a decomposition of K_{m+4} so let $n = m+4$ and $s = (n-1)/2$. Let D be a decomposition of K_m that contains all the cycles of Δ except one of the 6-cycles which we may assume is $C = [1,2,3,4,5,6]$. Label the other cycles $C_{s+1}, \ldots, C_{s+p+q+r-1}$. D also contains s path-graphs that contain the edges of C and also isolated vertices. If $m = 9$, then $s = 6$ and the path-graphs are

$$\begin{aligned} P_1 &= (1,2),6,7 \\ P_2 &= (2,3),1,7 \\ P_3 &= (3,4),2,8 \\ P_4 &= (4,5),3,8 \\ P_5 &= (5,6),4,9 \\ P_6 &= (1,6),5,9. \end{aligned}$$

If $m = 13$, then there are two further path-graphs

$$\begin{aligned} P_7 &= 10,11,12,13 \\ P_8 &= 10,11,12,13. \end{aligned}$$

For $m \ge 17$, there are further path-graphs $P_9, \ldots, P_s$, where, for $1 \le i \le (s-8)/2$,

$$P_{7+2i} = P_{8+2i} = 4i+10, 4i+11, 4i+12, 4i+13.$$

As the cycles of D form a decomposition of $K_m - C$, $v \in V(K_m) \setminus C$ will be in $s-2$ of them; v is also in 2 of the path-graphs. If $v \in C$, then it is in only $s-3$ of the cycles of

D, but is in 3 of the path-graphs. As every vertex is in s of the graphs of D, it is an odd $(s, p+q+r-1)$-decomposition of K_m. Use D to apply Theorem 2 with n, m and s as defined and $t = p+q+r-1$. The decomposition of K_n obtained contains all the cycles of D and also cycles $C_1, \ldots, C_s$ that are supergraphs of the path-graphs $P_1, \ldots, P_s$, and C_i, $1 \le i \le s$, has length $|V(P_i)|+n-m = 8$. The decomposition of K_n that is obtained contains p 4-cycles, $q-1$ 6-cycles and $r+s$ 8-cycles. Thus we can obtain a decomposition of K_n, $n = 1 \bmod 4$, $n \ge 13$, into p' 4-cycles, q' 6-cycles and r' 8-cycles from a cycle decomposition of K_{n-4} whenever $r' \ge s$.

Example 5. An example for complete graphs of even order. Let Δ be a decomposition of $K_m - I_m$, $m \ge 8$ even, into p 4-cycles, q 6-cycles and r 8-cycles. We are going to find a decomposition of K_{m+4} so let $n = m+4$, $s = (n-2)/2$ and $t = p+q+r-1$. Let one of the 4-cycles be $C = [1,2,3,4]$. Let D be a decomposition of K_m that contains the cycles of $\Delta - C$ (labelled $C_{s+1}, \ldots, C_{s+t}$), a matching $P_0 = I_m$ and s path-graphs that contain the edges of C. If $m = 8$, the path-graphs are

$$\begin{aligned} P_1 &= (1,2),5,6 \\ P_2 &= (2,3),5,6 \\ P_3 &= (3,4),7,8 \\ P_4 &= (1,4),7,8 \\ P_5 &= 1,2,3,4. \end{aligned}$$

If $m = 10$, then $P_2, \ldots, P_5$ are as above and

$$\begin{aligned} P_1 &= (1,2),9,10 \\ P_6 &= 5,6,9,10. \end{aligned}$$

For $m \ge 12$, let $P_1 = (1,2), m-1, m$, $P_2, \ldots, P_5$ be as above and

$$\begin{aligned} P_6 &= 5,6,9,10 \\ P_7 &= 9,10,11,12 \\ P_8 &= 11,12,13,14 \\ \vdots \; & \; \vdots \; \; \vdots \\ P_s &= m-3, m-2, m-1, m. \end{aligned}$$

Notice that D is an even (s,t)-decomposition of K_m (it is easy to check that every vertex is in P_0 and s of the other graphs in D). Thus from D, a cycle decomposition of $K_n - I_n$ is obtained by applying Theorem 2 with n, m, s and t as defined. The decomposition of $K_n - I_n$ obtained contains cycles $C_1, \ldots, C_s$ of length, for $1 \le i \le s$, $|V(P_i)| + n - m = 8$. Therefore it contains $p-1$ 4-cycles, q 6-cycles and $r+s$ 8-cycles, and we note that we can obtain a decomposition of $K_n - I_n$, $n \ge 12$ even, into p' 4-cycles, q' 6-cycles and r' 8-cycles from a cycle decomposition of $K_{n-4} - I_{n-4}$ whenever $r \ge s$.

Proof of Theorem 2: Necessity: for $1 \le i \le s$, C_i contains the edges of P_i plus at most $2(n-m)$ edges from $E(K_n) \setminus E(K_m)$. As it has length $|V(P_i)| + n - m$ we have

$$|E(P_i)| + 2(n-m) \ge |V(P_i)| + n - m.$$

Rearranging, (1) is obtained. Similarly, I_n contains the edges of P_0 plus at most $n-m$ edges from $E(K_n) \setminus E(K_m)$. As I_n has $n/2$ edges,

$$|E(P_0)| + n - m \geq n/2.$$

Rearranging, (2) is obtained.

Sufficiency: to simplify the presentation we will prove only the (slightly trickier) case where n is even. Thus $s = (n-2)/2$. Let the vertices of K_m be $v_1, \ldots, v_m$.

First consider the case $m = n-1$. From (1) and (2) we find that, for $1 \leq i \leq s$,

$$\begin{aligned} |E(P_i)| &\geq |V(P_i)| - 1, \text{ and} \\ |E(P_0)| &\geq n/2 - 1. \end{aligned}$$

In fact, we must have equality in each case since P_0 is a matching on $n-1$ vertices and, for $1 \leq i \leq s$, P_i is acyclic. Thus each P_i, $1 \leq i \leq s$, must be a single path and P_0 contains $n/2 - 1$ independent edges and an isolated vertex. Each vertex has degree $n-2$ in K_{n-1}, is in $s = (n-2)/2$ of the subgraphs $P_1, \ldots, P_s, C_{s+1}, \ldots, C_{s+t}$ and has degree at most two in each of these subgraphs. Thus the vertex that has degree 0 in P_0 must have degree 2 in each of the other subgraphs that contain it, and each vertex of degree 1 in P_0, must have degree 1 in one of the other subgraphs that contain it and degree 2 in the rest; that is, it must be the endvertex of precisely one of the paths P_i, $1 \leq i \leq s$. Therefore we obtain the cycle decomposition of K_n from D, the (s,t)-decomposition of K_{n-1}, by adding edges (v_j, v_n), $1 \leq j \leq n-1$, to the subgraphs in the following way. If v_j is an endvertex in P_i, then the new edge (v_j, v_n) is placed in the subgraph P_i. Hence P_i becomes a cycle of length $|V(P_i)| + 1$. Finally, if v_j is the isolated vertex in P_0, then (v_j, v_n) is the additional edge required to form the one-factor I_n.

Now we show that if $m < n-1$, then D can be extended to D', an even (s,t)-decomposition of K_{m+1}, so that (1) and (2) are satisfied with m replaced by $m+1$. By repeating this argument a finite number of times an (s,t)-decomposition of K_{n-1} that satisfies (1) and (2) with m replaced by $n-1$ can be found.

To obtain D', a new vertex v_{m+1} is added to K_m. It must be joined to each vertex of K_m by one edge and each of these m additional edges must be added to exactly one of the path-graphs or the matching of D. Note that we require that v_{m+1} is in all s of the path-graphs, so it must be added as an isolated vertex to any path-graph that has been given no new edges.

We need a way to decide which subgraph each new edge should be placed in. Construct a bipartite multigraph B with vertex sets $\{P'_0, \ldots, P'_s\}$ and $\{v'_1, \ldots, v'_m\}$. For $1 \leq i \leq s$, $1 \leq j \leq m$, if $v_j \in P_i$, then join P'_i to v'_j by $2 - d_{P_i}(v_j)$ edges. Also join P'_0 to v_j by $1 - d_{P_0}(v_j)$ edges. In fact, we think of B as being constructed as follows: for $1 \leq i \leq s$ join P'_i to v'_j by two edges if $v_j \in P_i$ and join P'_0 to v_j by one edge; then for each edge (v_j, v_k) in P_i, $0 \leq i \leq s$, delete the edges (P'_i, v'_j) and (P'_i, v'_k).

If v_j is in x of the cycles in D, then it is in $s - x$ of the path-graphs; it is also in the matching P_0. As it is incident with $2x$ edges in the cycles, it is incident with $m-1-2x$ edges in the matching and path-graphs. When B is constructed, we begin by placing $2(s-x)+1$ edges at v'_j. For $0 \leq i \leq s$, for each edge incident at v_j in P_i, we delete an edge (P'_i, v'_j) in B. Therefore

$$d_B(v'_j) = 2(s-x) + 1 - (m-1-2x) = n - m. \tag{3}$$

When we construct B we first place $2|V(P_i)|$ edges at P_i', $1 \le i \le s$. Then for each edge (v_j, v_k) in P_i, we delete two of these edges: (P_i', v_j') and (P_i', v_k'). Thus, by (1), for $1 \le i \le s$,

$$d_B(P_i') = 2(|V(P_i)| - |E(P_i)|) \le 2(n-m). \tag{4}$$

Using a similar argument, by (2),

$$d_B(P_0') = |V(P_0)| - |E(P_0)| \le n - m. \tag{5}$$

We need the following: a set $\mathcal{F}$ of sets is a *laminar* set if, for all $X, Y \in \mathcal{F}$, either $X \subseteq Y$, or $Y \subseteq X$ or $X \cap Y = \emptyset$; we say $x \approx y$ if $\lfloor y \rfloor \le x \le \lceil y \rceil$ (note that the relation is not symmetric).

Lemma 3 [8] *If $\mathcal{F}$ and $\mathcal{G}$ are laminar sets of subsets of a finite set M and h is a positive integer then there exists a set $L \subseteq M$ such that*

$$|L \cap X| \approx |X|/h \text{ for every } X \in \mathcal{F} \cup \mathcal{G}.$$

We construct two laminar sets $\mathcal{F}$ and $\mathcal{G}$ which contain subsets of $E(B)$. Let $\mathcal{F}$ contain sets $P_0^*, \ldots, P_s^*$, where P_i^* contains the set of all edges incident with P_i' in B. Also if v_{j_1} and v_{j_2} are endvertices of a path in P_i, then let $\{(P_i', v_{j_1}'), (P_i', v_{j_2}')\}$ be a set in $\mathcal{F}$ (call these endvertex-sets). Let $\mathcal{G}$ contain sets $v_1^*, \ldots, v_m^*$, where v_j^* contains the set of all edges incident with v_j' in B.

Apply Lemma 3 with $M = E(B)$ and $h = n - m$ to obtain a set of edges L that, by (3), (4) and (5), contains exactly one edge incident with v_j', $1 \le j \le m$, at most two edges incident with P_i', $1 \le i \le s$, and at most one edge incident with P_0. Also L contains at most one edge from each endvertex-set.

Now we extend D to D'. For $1 \le j \le n$, if (P_i', v_j') is in L, then (v_{m+1}, v_j) is placed in P_i. Then v_{m+1} is added as an isolated vertex to any P_i to which no new edges have been added. Since L contains exactly one edge incident with each v_j, each new edge is placed in exactly one subgraph. There is only an edge (P_i', v_j'), $1 \le i \le s$, $1 \le j \le m$, in B if v_j has degree less than 2, so after the new edges are added v_j has degree at most 2 in P_i. Since L contains at most two edges incident with P_i, $1 \le i \le s$, v_{m+1} has degree at most 2 in each P_i, $1 \le i \le s$. As L contains at most one edge from each endvertex-set, v_{m+1} cannot have been joined to both ends of a path in P_i (thus creating a cycle). Therefore P_i, $1 \le i \le s$, is still a path-graph. By a similar argument, P_0 is still a matching.

We must check that (1) and (2) remain satisfied with m replaced by $m+1$. First (1): note that $|V(P_i)|$ increases by one (as the new vertex is adjoined to every path-graph) and $E(P_i)$ increases by at most two. If initially we have

$$n - m - 2 \ge |V(P_i)| - |E(P_i)|,$$

then clearly (1) remains satisfied. If

$$n - m - 1 = |V(P_i)| - |E(P_i)|,$$

then, arguing as for (4), $d_B(P_i') = 2(|V(P_i)| - |E(P_i)|) = 2(n-m) - 2 \ge n - m$ (since $n - m \ge 2$). So L contains at least one edge incident with P_i' and at least one edge is added to P_i and (1) remains satisfied. If

$$n - m = |V(P_i)| - |E(P_i)|,$$

then $d_B(P_i') = 2(n-m)$, and L contains two edges incident with P_i' and hence two edges are added to P_i and (1) remains satisfied.

Finally, if initially we have

$$|E(P_0)| - 1 \geq m - n/2$$

then (2) remains satisfied. If

$$|E(P_0)| = m - n/2$$

then $d_B(P_0') = n - m$, and L contains an edge incident with P_0' and hence an edge is added to P_0 and (2) remains satisfied. □

3 Cycle decompositions of complete bipartite graphs

In this section we prove a result on cycle decompositions of $K_{m,n}$ that will be useful in the proof of Theorem 1.

Theorem 4 *The complete bipartite graph $K_{m,n}$ can be decomposed into p 4-cycles, q 6-cycles and r 8-cycles if and only if*

1. *m and n are even,*
2. *no cycle has length greater than $2\min\{m,n\}$,*
3. *$mn = 4p + 6q + 8r$,*
4. *if $m = n = 4$, then $r \neq 1$.*

Proof: Necessity: the first condition is necessary since each vertex must have even degree if the graph is to be decomposed into cycles; the second because half the vertices in each cycle must belong to same independent set of $K_{m,n}$; the third because each edge of the graph must be in exactly one of the cycles; and the fourth because $K_{4,4}$ cannot be decomposed into two 4-cycles and one 8-cycle (since $K_{4,4} - C_8 = C_8 \neq 2C_4$).

Sufficiency: when seeking a cycle decomposition of $K_{m,n}$ we will assume that cycle decompositions of $K_{m,n'}$, $n' < n$ have been found. Let us see how this assumption will help us. Suppose that $n = n_1 + n_2 \cdots + n_a$ where each n_i is a positive even integer. Then $K_{m,n}$ is the union of $K_{m,n_1}, K_{m,n_2}, \ldots, K_{m,n_a}$. From decompositions of K_{m,n_i}, $1 \leq i \leq a$, into p_i 4-cycles, q_i 6-cycles and r_i 8-cycles where

$$\begin{aligned} p &= p_1 + p_2 + \cdots + p_a, \\ q &= q_1 + q_2 + \cdots + q_a, \text{ and} \\ r &= r_1 + r_2 + \cdots + r_a, \end{aligned}$$

we can find a decomposition of $K_{m,n}$ into p 4-cycles, q 6-cycles and r 8-cycles. So to find a decomposition of $K_{m,n}$ all we have to do is to assign the required cycles to the smaller graphs, making sure that, for $i \leq a-1$, the cycles assigned to K_{m,n_i} have a total of mn_i edges (the $i = a$ case takes care of itself once the others have been checked). We must also be sure to assign only 4-cycles to K_{m,n_i} if $n_i = 2$ and not to assign two 4-cycles and one 8-cycle to $K_{4,4}$.

Notation: let the vertex sets of $K_{m,n}$ be $\{a, b, c, \ldots\}$ and $\{1, 2, 3, \ldots\}$.

We note that the cases where all the cycles have the same length were proved by Sotteau [11]. We prove the remaining cases. We assume that $m \leq n$.

Case 1. $m = 2$. We must have $p = n/2$, $q = 0$ and $r = 0$ (by conditions 2 and 3). Observe that $K_{2,n}$ is the union of $(n/2)$ 4-cycles.

Case 2. $m = 4, n = 4$. We have noted that $K_{4,4}$ cannnot be decomposed into two 4-cycles and an 8-cycle. The only remaining case with cycles not all the same length is $p = 1$, $q = 2$ and $r = 0$. We describe a decomposition: $[b, 3, d, 4]$, $[a, 1, b, 2, c, 3]$, $[a, 2, d, 1, c, 4]$.

Case 3. $m = 4$, $n = 6$. As there are 24 edges, we require that $24 - 6q = 0 \bmod 4$. Therefore $q \in \{0, 2, 4\}$. If $q = 4$ then the cycles all have the same length.

Suppose that $q = 0$. If $p = 2$ and $r = 2$ then we obtain the decomposition of $K_{4,6}$ by combining a decomposition of $K_{4,2}$ into 4-cycles with a decomposition of $K_{4,4}$ into 8-cycles. We describe a decomposition for $p = 4$ and $r = 1$: $[a, 2, c, 6]$, $[a, 3, b, 5]$, $[b, 4, d, 6]$, $[c, 1, d, 5]$, $[a, 1, b, 2, d, 3, c, 4]$.

Suppose that $q = 2$. If $p = 3$ then we obtain the decomposition of $K_{4,6}$ by combining a decompostion of $K_{4,4}$ into one 4-cycle and two 6-cycles with a decomposition of $K_{4,2}$ into 4-cycles. We describe a decomposition for $p = 1$ and $r = 1$: $[b, 4, c, 6]$, $[a, 2, d, 5, b, 3]$, $[a, 5, c, 1, d, 6]$, $[a, 1, b, 2, c, 3, d, 4]$.

Case 4. $m = 4$, $n \geq 8$. As $K_{4,n}$ has at least 32 edges, either $p \geq 2$, $q \geq 4$ or $r \geq 2$ (else $4p + 6q + 8r < 32$).

If $p \geq 2$, then we obtain the decomposition of $K_{4,n}$ by combining a decomposition of $K_{4,2}$ into 4-cycles with a decomposition of $K_{4,n-2}$.

Suppose that $p < 2$ and $q \geq 4$. If $n = 8$ then $p = 0$, $q = 4$ and $r = 1$. We describe a decomposition: $[a, 1, b, 3, c, 5]$, $[a, 2, b, 5, d, 7]$, $[a, 6, d, 2, c, 8]$, $[b, 6, c, 4, d, 8]$, $[a, 3, d, 1, c, 7, b, 4]$. If $n \geq 10$ then we obtain the decomposition of $K_{4,n}$ by combining a decomposition of $K_{4,6}$ into 6-cycles with a decomposition of $K_{4,n-6}$. (Notice that we are not requiring $K_{4,4}$ to be decomposed into two 4-cycles and one 8-cycle since $p < 2$.)

If $p < 2$ and $q < 4$, then $r \geq 2$ and we obtain the decomposition of $K_{4,n}$ by combining a decomposition of $K_{4,4}$ into 8-cycles with a decomposition of $K_{4,n-4}$.

Case 5. $m = 6$, $n = 6$. There are 36 edges so we require that $36 - 6q = 0 \bmod 4$. Therefore $q \in \{0, 2, 4, 6\}$. If $q = 6$ then the cycles all have the same length.

If $p \geq 3$ then we obtain the decomposition of $K_{6,6}$ by combining a decomposition of $K_{2,6}$ into 4-cycles with a decomposition of $K_{4,6}$.

If $p < 3$ and $q = 0$, the only possibility is that $p = 1$ and $r = 4$ for which we describe a decomposition: $[a, 1, e, 2]$, $[a, 3, b, 4, c, 1, d, 5]$, $[a, 4, d, 3, f, 1, b, 6]$, $[b, 2, f, 4, e, 6, c, 5]$, $[c, 2, d, 6, f, 5, e, 3]$.

If $p < 3$ and $q = 2$, then there are two possibilities: we describe decompositions for each. First, $p = 2$ and $r = 2$: $[a, 1, c, 4]$, $[a, 2, d, 5]$, $[b, 1, d, 3, e, 4]$, $[b, 3, f, 2, e, 6]$, $[a, 3, c, 5, f, 4, d, 6]$, $[b, 2, c, 6, f, 1, e, 5]$. And $p = 0$ and $r = 3$: $[a, 1, b, 2, c, 3]$, $[a, 2, d, 3, e, 4]$, $[a, 5, d, 1, c, 4, f, 6]$, $[b, 3, f, 1, e, 5, c, 6]$, $[b, 4, d, 6, e, 2, f, 5]$.

If $p < 3$ and $q = 4$, then $p = 1$ and $r = 1$. We describe the decomposition: $[a, 1, b, 2]$, $[a, 3, c, 1, d, 4]$, $[a, 5, c, 2, e, 6]$, $[b, 3, f, 2, d, 5]$, $[b, 4, e, 5, f, 6]$, $[c, 4, f, 1, e, 3, d, 6]$.

Case 6. $m = 6$, $n = 8$. There are 48 edges so either $p \geq 3$, $q \geq 4$ or $r \geq 3$. If $p \geq 3$ then we obtain the decomposition of $K_{6,8}$ by combining a decomposition of $K_{6,2}$ into 4-cycles with a decomposition of $K_{6,6}$. If $p < 3$ then either $q \geq 4$ or $r \geq 3$ and we obtain the decomposition of $K_{6,8}$ by combining a decomposition of $K_{6,4}$ into either 6-cycles or 8-cycles with another decomposition of $K_{6,4}$.

Case 7. $m = 8$, $n = 8$. If $4p + 8r \geq 32$ then we obtain the decomposition of $K_{8,8}$ by combining a decomposition of $K_{4,8}$ into 4-cycles and 8-cycles with another decomposition of $K_{4,8}$. If $4p + 8r < 32$ then $6q > 32$ and so $q \geq 6$.

If $r \geq 1$, then we obtain the decomposition of $K_{8,8}$ by combining a decomposition of $K_{4,8}$ into four 6-cycles and one 8-cycle with another decomposition of $K_{4,8}$.

Suppose that $r = 0$. If $p \geq 4$ then we obtain the decomposition of $K_{8,8}$ by combining a decomposition of $K_{2,8}$ into 4-cycles with a decomposition of $K_{6,8}$. We describe a decomposition for the remaining case, $p = 1$, $q = 10$: $[a, 1, b, 2]$, $[a, 3, b, 4, d, 6]$, $[a, 4, c, 1, e, 7]$, $[a, 5, d, 1, f, 8]$, $[b, 5, f, 2, c, 7]$, $[b, 6, e, 4, g, 8]$, $[c, 3, d, 7, g, 5]$, $[c, 6, h, 3, e, 8]$, $[d, 2, e, 5, h, 8]$, $[f, 3, g, 1, h, 4]$, $[f, 6, g, 2, h, 7]$.

Case 8. $m \geq 6$, $n \geq 10$. If $4p + 8r \geq 4m$ then we obtain the decomposition of $K_{m,n}$ by combining a decomposition of $K_{m,4}$ into 4-cycles and 8-cycles with a decomposition of $K_{m,n-4}$. If $4p+8r < 4m$ then $6q \geq 6m$ (since $4p+6q+8r = nm \geq 10m$) and we obtain the decomposition of $K_{m,n}$ by combining a decomposition of $K_{m,6}$ into 6-cycles with a decomposition of $K_{m,n-6}$. $\square$

4 Proof of Theorem 1

The necessity of the conditions is clear.

Sufficiency: as we remarked in the Introduction, all possible cycle decompositions of K_n have been found for $n \leq 10$ [9]. For $n > 10$, we assume that cycle decompositions for $K_{n'}$, $n' < n$, are known. Then we either apply Theorem 2 to extend a decomposition of some $K_{n'}$, or, as we did for complete bipartite graphs in the previous section, we consider K_n as the union of several edge-disjoint subgraphs and assign the cycles required in the decomposition of K_n to the subgraphs (decompositions of which we will be able to assume are known).

First suppose that n is odd. Note that if $n = 3 \bmod 4$, K_n has an odd number of edges and cannot have a decompisition into cycles of even length.

Case 1. $n = 13$. We require a decomposition of K_{13} into p 4-cycles, q 6-cycles and r 8-cycles. Note that

$$4p + 6q + 8r = |E(K_{13}| = 78.$$

Thus q is odd since $78 \neq 0 \bmod 4$. In Section 2, we saw that we could find a decomposition of K_{13} by extending a decomposition of K_9 if $r \geq 6$ (Example 4) or by extending a decomposition of K_{10} if $q \geq 5$ (Example 3). We may now assume that $q \in \{1, 3\}$ and $r \leq 5$, which implies that $4p \geq 78 - (3 \times 6) - (5 \times 8) = 20$, that is, $p \geq 5$.

Consider K_{13} as the union of K_5, K_9 and $K_{8,4}$ where K_5 is defined on the vertex set $\{1, \ldots, 5\}$, K_9 on the vertex set $\{5, \ldots, 13\}$, and $K_{4,8}$ on the vertex sets $\{1, \ldots, 4\}$ and $\{6, \ldots, 13\}$. Let $C = [1, 6, 2, 7]$. Let $H_1 = K_5 \cup C$ and $H_2 = K_{8,4} - C$. Note that H_1 is the union of two 4-cycles and a 6-cycle. K_{13} is the union of K_9, H_1 and H_2. For the remaining cases, we assign the cycles required in the decomposition of K_{13} to these three subgraphs. (A decomposition of H_2 is found by finding a decomposition of $K_{8,4}$ that contains, as well as the required cycles, a further 4-cycle which can be labelled $[1, 6, 2, 7]$ and discarded.) Two

4-cycles and a 6-cycle are assigned to H_1. There are at most two further 6-cycles which are assigned to K_9. Up to three 8-cycles are assigned to H_2; any remaining 8-cycles (there are at most two more) are assigned to K_9. This only leaves some 4-cycles to be assigned, and clearly the number of edges not accounted for in H_2 and K_9 is, in both cases, positive and equal to 0 mod 4.

Case 2. $n = 17$. We require a decomposition of K_{17} into p 4-cycles, q 6-cycles and r 8-cycles. Note that

$$4p + 6q + 8r = |E(K_{17})| = 136.$$

If $r \geq (n-1)/2 = 8$, then, as seen in Example 4, we can apply Theorem 2 to obtain the decomposition of K_{17} from a decomposition of K_{13}.

For the remaining cases, let $K_{17} = K_9 \cup K_9 \cup K_{8,8}$. We will assign the required cycles to these three subgraphs. Suppose that $4p + 8r \geq 64$. Then we can assign all of the 8-cycles (since $r < 8$) and some 4-cycles to $K_{8,8}$ so that the assigned cycles have precisely 64 edges. There remain to be assigned some 4-cycles and 6-cycles which have a total of 72 edges. Thus either $4p \geq 36$ or $6q \geq 36$ and we can assign cycles all of the same length to a K_9. We assign the remaining cycles to the other K_9.

If $4p + 8r < 64$, then $6q > 136 - 64 = 72$. We can assign six 6-cycles to each K_9 and the remaining cycles to $K_{8,8}$.

Case 3. $n \geq 21$ odd. We require a decomposition of K_n into p 4-cycles, q 6-cycles and r 8-cycles. If $r \geq (n-1)/2$, then, as seen in Example 4, we can apply Theorem 2 to obtain the decomposition of K_n from a decomposition of K_{n-4}. Otherwise $r < (n-1)/2$. Let $K_n = K_{n-12} \cup K_{13} \cup K_{n-13,12}$. We assign the required cycles to these subgraphs.

Suppose that $4p + 8r \geq |E(K_{n-13,12})|$. Then we can assign all of the 8-cycles and some 4-cycles to $K_{n-13,12}$ (since $8r < 4(n-1) < 12(n-13)$ for $n \geq 21$). We are left with 4-cycles and 6-cycles to assign to K_{n-12} and K_{13}. We can assign cycles all of the same length to the smaller of these graphs (which is either K_9 or K_{13}—both have a number of edges equal to 0 mod 4 and 0 mod 6) and the remaining cycles to the other.

If $4p+8r < |E(K_{n-13,12})|$, then $6q \geq |E(K_{n-12})| + |E(K_{13})|$. We cannot simply assign 6-cycles to K_{n-12} and K_{13} however, since $|E(K_{n-12})|$ is not equal to 0 mod 6 for all $n = 1$ mod 4. If $6q \geq |E(K_{n-13,12})| + |E(K_{13})|$, then we can assign 6-cycles to $K_{n-13,12}$ and K_{13} and any remaining cycles to K_{n-12}. If $6q < |E(K_{n-13,12})| + |E(K_{13})|$ then $4p+8r > |E(K_{n-12})| > 16$ (as $n \geq 21$). One of the following must be true.

$$\begin{aligned} |E(K_{n-12})| &= 0 \bmod 6 \\ |E(K_{n-12})| + 8 &= 0 \bmod 6 \\ |E(K_{n-12})| + 16 &= 0 \bmod 6 \end{aligned}$$

We assign 6-cycles to K_{13} and to K_{n-12}, except that to K_{n-12} we also assign 4-cycles and 8-cycles with a total of 8 or 16 edges if the number of edges in K_{n-12} is 2 mod 6 or 4 mod 6 respectively. The remaining edges are assigned to $K_{n-13,12}$.

Case 4. $n = 12$. We require a decomposition of $K_{12} - I_{12}$ into p 4-cycles, q 6-cycles and r 8-cycles. Then

$$4p + 6q + 8r = |E(K_{12} - I_{12})| = 60,$$

and so q is even. In Example 5, we saw that if $r \geq (n-2)/2 = 5$, we can use Theorem 2 to extend a decomposition of K_8 to obtain the required cycle decomposition of $K_{12} - I_{12}$. Otherwise let $K_{12} - I_{12} = (K_6 - I_6) \cup (K_6 - I_6) \cup K_{6,6}$. We will assign the required cycles to these subgraphs.

Note that $4p + 6q \geq 60 - 32 = 28$ (since $r \leq 4$). If $q = 0$, then $p \geq 7$ and we can assign 4-cycles to each K_6. If $q = 2$, we assign the two 6-cycles to one K_6 and 4-cycles to the other K_6. If $q \geq 4$, we assign 6-cycles to each K_6. In each case the remaining cycles are assigned to $K_{6,6}$.

Case 5. $n \geq 14$ even. We require a decomposition of $K_n - I_n$ into p 4-cycles, q 6-cycles and r 8-cycles. In Example 5, we saw that if $r \geq (n-2)/2$, we can use Theorem 2 to extend a decomposition of K_{n-4} to find the required cycle decomposition of $K_n - I_n$. Otherwise let $K_n - I_n = (K_6 - I_6) \cup (K_{n-6} - I_{n-6}) \cup K_{6,n-6}$. We will assign the required cycles to these subgraphs.

We have

$$\begin{aligned} 4p + 6q &= |E(K_n - I_n)| - 8r \\ &\geq n(n-2)/2 - 4(n-4) \\ &= (n^2 - 10n + 32)/2 \\ &\geq 44, \end{aligned}$$

as $n \geq 14$. Therefore $4p \geq 22$ or $6q \geq 22$ and we can assign cycles all of length 4 or all of length 6 to $K_6 - I_6$. Note that the number of edges in $K_{6,n-6}$ is 0 mod 12. If $q \geq n-6$ we assign only 6-cycles to $K_{6,n-6}$. Otherwise we assign all the 6-cycles to $K_{6,n-6}$ if q is even, or all but one of them if q is odd. Then the number of remaining edges is also 0 mod 12 so we can assign as many 4-cycles as necessary. All remaining cycles are assigned to K_{n-6}. □

References

[1] P. Adams, D.E. Bryant and A. Khodkar, 3, 5-cycle decompositions, J. Combin. Designs **6** (1998), 91–110.

[2] P. Adams, D.E. Bryant, A. Khodkar, On Alspach's conjecture with two even cycle lengths, Discrete Math. **223** (2000), 1–12.

[3] B. Alspach, Research Problems, Problem 3, Discrete Math. **35** (1981), 333.

[4] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, J. Combin. Theory B **81** (2001), 77–99.

[5] P.N. Balister, On the Alspach conjecture, Comb. Probab. Comput. **10** (2001), 95–125.

[6] D.E Bryant, A. Khodkar and H.L. Fu, (m, n)-cycle systems, J. Statist. Planning & Inference **74** (1998), 365–370.

[7] K. Heinrich, P. Horak and A. Rosa, On Alspach's conjecture, Discrete Math. **77** (1989), 97–121.

[8] C. St. J. A. Nash-Williams, Amalgamations of almost regular edge-colourings of simple graphs, J. Combin. Theory B, **43** (1987), 322-342.

[9] A. Rosa, Alspach's conjecture is true for $n \leq 10$, Math. Reports, McMaster University.

[10] M. Šajna, Cycle decompositions III: Complete graphs and fixed length cycles, J. Combin. Designs **10** (2002), 27–78.

[11] D. Sotteau, Decompositions of $K_{m,n}$ ($K^*_{m,n}$) into cycles (circuits) of length $2k$ J. Combin. Theory B **30** (1981), 75–81.