Amalgamations of connected k-factorizations

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Abstract

In this paper, necessary and sufficient conditions are found for a graph with exactly one amalgamated vertex to be the amalgamation of a k-factorization of K_{kn+1} in which each k-factor is connected. From this result, necessary and sufficient conditions for a given edge-coloured K_t to be embedded in a connected k-factorization of K_{kn+1} are deduced.

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1 Introduction

Over the past 30 years there has been substantial interest in finding partitions P of $E(G)$ with the property that each element of P induces a graph that is isomorphic to a given fixed graph H; these are known as *H-decompositions* of G. For example, the existence problem when $G = K_n$, that is, the problem of finding the values of n for which there exists an H-decomposition of K_n, has been solved when H is a cycle [1, 17], a star [18], a path [19], and K_k when $k \in \{3, 4\}$ [5, 9]; for a survey, see [11]. The problem of embedding an H-decomposition of G into an H-decomposition of K_n has also been addressed, as has a multigraph version of this problem; see [3, 4, 8, 10, 12, 13] for example.

Related, but substantially different, problems that have been considered more recently are the existence and embedding problems when H is required to be a *spanning* subgraph of G; so now the size of H increases with the size of G. Not only have these problems been of interest in their own right, but they have been attacked by using a powerful proof technique known as amalgamations of graphs. It is this technique that we use here. Exploring these problems has also introduced graph theoretic issues and methods into what was originally largely of design theoretic interest. For example Andersen, Hilton, Rodger and Wantland [2, 16] relied heavily on edge-colourings of graphs to prove the following neat embedding result.

Theorem 1.1 *An n-edge-coloured K_t, with colour classes $G(1), G(2) \ldots, G(n)$, can be embedded in an n-edge-coloured K_{kn+1} in which each colour class is a k-factor if and only if*

(1) $d_{G(i)}(v) \leq k$ for each $v \in V(K_t)$ and for $1 \leq i \leq n$,

(2) $\min\{|E(G(i))| : 1 \leq i \leq n\} \geq (2t - kn - 1)k/2$, and

(3) if k is odd, then n is odd.

Hilton [6] went on in the case where $k = 2$ to require that each k-factor be connected, that is, each k-factor is a hamilton cycle (see also [7] for a multipartite version of the same problem). Rodger and Wantland [16] generalized Hilton's result to 2-edge-connected k-factorizations (this also follows from a result of Nash-Williams [14]).

Theorem 1.2 *An n-edge-coloured K_t, with colour classes $G(1), G(2) \ldots, G(n)$, can be embedded in an n-edge-coloured K_{kn+1} in which each colour class is a 2-edge-connected k-factor if and only if*

(1) conditions (1-3) of Theorem 1.1 are satisfied,

(2) $\sum_{v \in V(C)} d_{G(i)}(v) \leq k|V(C)| - 2$ for each component C of $G(i)$ and for $1 \leq i \leq n$, and

(3) if for $1 \leq i \leq n$, there is a cutedge e in a component C of $G(i)$, then the components C_1 and C_2 of $C - e$ each contain a vertex of degree less than k in $G(i)$.

So if each k-factor is required to be m-edge-connected, then Theorems 1.1 and 1.2 solve the embedding problem where $m = 0$ and $m = 2$ respectively. However the case where $m = 1$ and k is odd has remained open for some time, proving to be a tough nut to crack! (If k is even, then connected and 2-edge-connected are equivalent notions.) One of the main results of this paper is to solve the embedding problem when $m = 1$ (see Theorem 3.1).

Each of Theorems 1.1 and 1.2 can be proved using *amalgamations*, and they are used to prove Theorem 3.1. We now introduce this structure.

Let G be an edge-coloured graph on the set of colours $\{1, 2, \ldots n\}$, and let P be a partition of $V(G)$. The *P-amalgamation* of G is the n-edge-coloured graph with vertex set P, in which $p \in P$ is incident with y loops coloured i if and only if there are y loops or edges in G that join vertices in p, and in which the number of edges coloured i joining $p_1, p_2 \in P, p_1 \neq p_2$, is the number of edges coloured i in G that join vertices in p_1 to vertices in p_2. We say that H is an *amalgamation* of G if it is the P-amalgamation of G for some partition P of $V(G)$.

An *outline k-factorization* of K_{kn+1} is an n-edge-coloured graph H and an associated function $f = f_H : V(H) \to \mathbb{N}$ that satisfies the properties:

$P(a):$ for each v, $w \in V(H)$, the number of edges joining v and w is $f(v)f(w)$,

$P(b):$ for each $v \in V(H)$, the number of loops incident with v is $\binom{f(v)}{2}$,

$P(c):$ for $1 \leq i \leq n$, each $v \in V(H)$ is incident with $kf(v)$ edges coloured i (counting loops as 2 edges), and

$P(d):$ $\displaystyle\sum_{v \in V(H)} f(v) = kn+1.$

It is not hard to see that any P-amalgamation of any k-factorization of K_{kn+1} is an outline k-factorization of K_{kn+1}. It then becomes a captivating question to ask if the converse is also true: it is!

Theorem 1.3 [2, 16] *Each outline k-factorization of K_{kn+1} is the amalgamation of a k-factorization of K_{kn+1}.*

It turns out that Theorem 1.1 follows easily from Theorem 1.3, and so perhaps not too surprisingly, Theorem 1.2 follows from the following result of Nash-Williams [14], and of Rodger and Wantland [16].

Theorem 1.4 *Each outline k-factorization of K_{kn+1} in which each colour class is 2-edge-connected is the amalgamation of a k-factorization of K_{kn+1} in which each k-factor is 2-edge-connected.*

The other main result (see Theorem 2.3) of this paper is a first attempt to provide the companion result to Theorems 1.3 and 1.4 in the case where each k-factor is only required to be *connected* (rather than 2-edge-connected).

Throughout the remainder of this paper we will use the following notation. The function f_H is usually represented by f. Let H be an outline k-factorization of K_{kn+1} on the vertex set $\{v_0, v_1, \ldots, v_{r-1}\}$. For $1 \leq i \leq n$ and for each $v \in V(H)$, let $H_i(v)$ and $H(i)$ be the subgraphs of $H - v$ and of H respectively that are induced by the edges coloured i, let $\omega_i(v)$ be the number of components in $H_i(v)$, let $C_{i,1}(v), C_{i,2}(v), \ldots, C_{i,\omega_i(v)}(v)$ be the components of $H_i(v)$, and let $L_i(v)$ be the set of loops coloured i that are incident with v in H. For $1 \leq i \leq n$, for each $v \in V(H)$, and for $1 \leq j \leq \omega_i(v)$, let $E_{i,j}(v)$ be the set of edges in H

coloured i that join vertices in $C_{i,j}(v)$ to v, and let $\epsilon_{i,j}(v) = |E_{i,j}(v)|$. For $1 \leq i \leq n$, let $E_i(v) = \bigcup_{j=1}^{\omega_i(v)} E_{i,j}(v)$, $\epsilon_i(v) = |E_i(v)|$ and $l_i(v) = |L_i(v)|$. For any undefined graph theoretic terms, see [20].

An outline k-factorization H of K_{kn+1} is said to be *tree-connected* if for $1 \leq i \leq n$:

$C(a):$ $H(i)$ is connected, and

$C(b):$ for all $v \in V(H), f(v) + \omega_i(v) \leq \epsilon_i(v) + l_i(v) + 1$.

We refer to the bound $C(b)$ as the *tree bound* for $H(i)$ (see the proof of Lemma 2.1 for the reason for this name). As Lemma 2.1 proves, requiring an outline k-factorization to be tree-connected is necessary for it to be the amalgamation of a connected k-factorization of K_{kn+1}, and as Theorem 2.3 proves it is sufficient in the case where $f(v) = 1$ for all vertices except possibly one. Even with this restriction, Theorem 2.3 is strong enough to obtain our embedding result, Theorem 3.1.

2 The Results

We begin by showing that tree-connected outline k-factorizations are natural associates of amalgamations of connected k-factorizations of K_{kn+1}. (Notice that a vertex v in an amalgamation of a graph G is formally a set of vertices in G; we use a lower case letter because it is more fruitfully thought of as a vertex rather than a set.)

Lemma 2.1 *Suppose H is a $\{v_0, \ldots, v_{r-1}\}$-amalgamation of a k-factorization of K_{kn+1} in which each k-factor is connected. Then H is a tree-connected outline k-factorization of K_{kn+1} with $f_H(v_z) = |v_z|$ for $0 \leq z \leq r-1$.*

Proof: It is easy to see (and it is shown in [2] and [16]) that H is an outline k-factorization of K_{kn+1}. Clearly $H(i)$ is connected. We need to show that the tree bound $C(b)$ is satisfied. Let the k-factors of K_{kn+1} be $F(1), F(2), \ldots, F(n)$. Let the vertex set of K_{kn+1} be W. Recall that v_z is a subset of W for $0 \leq z \leq r-1$. For each $v \in V(H)$, if $v = \{w_1, \ldots, w_{f(v)}\}$, then let

H_v be the $\{\{w_1\},\ldots,\{w_{f(v)}\},V(C_{i,1}(v)),\ldots,V(C_{i,\omega_i}(v))\}$-amalgamation of F. Since $F(i)$ is connected, $H_v(i)$ is connected for $1 \leq i \leq n$, so $\epsilon(H_v(i)) \geq f(v)+\omega_i(v)-1$ (the number of edges in a spanning tree on $f(v)+\omega_i(v)$ vertices). An edge in $H_v(i)$ joining a pair of vertices in $\{w_1,\ldots,w_{f(v)}\}$ corresponds to a loop coloured i on v in H, and an edge in $H_v(i)$ joining a vertex in $\{w_1,\ldots,w_{f(v)}\}$ to a vertex in $\{V(C_{i,1}(v)),\ldots,V(C_{i,\omega_i}(v))\}$ corresponds to an edge coloured i in H joining a vertex in $V(H)\setminus\{v\}$ to v. Therefore $\epsilon(H_v(i)) = l_i(v)+\epsilon_i(v)$, so $l_i(v)+\epsilon_i(v) \geq f(v)+\omega_i-1$. □

We need the following for our proof of Theorem 2.3. A set $\mathcal{F}$ of sets is a *laminar* set if for all $X, Y \in \mathcal{F}$, $X \subseteq Y$, $Y \subseteq X$ or $X \cap Y = \emptyset$. We say $x \approx y$ if $\lfloor y \rfloor \leq x \leq \lceil y \rceil$. Lemma 2.2 is from [14].

Lemma 2.2 *If $\mathcal{F}$ and $\mathcal{G}$ are laminar sets of subsets of a finite set M and h is a positive integer then there exists a set $J \subseteq M$ such that*

$$|J \cap X| \approx |X|/h \text{ for every } X \in \mathcal{F} \cup \mathcal{G}. \tag{1}$$

We are now ready to prove the first main result of this paper.

Theorem 2.3 *Let H be a tree-connected outline k-factorization of K_{kn+1} in which*

$$f(v) = 1 \text{ for all vertices } v \text{ in } G \text{ except possibly for one.}$$

Then H is an amalgamation of a connected k-factorization of K_{kn+1}.

Remark 1 If k is even, then connected and 2-edge-connected are equivalent, so Theorem 2.3 follows from Theorem 1.4 in this case.

Remark 2 In the following proof, the assumption that $f(v) = 1$ for all vertices except possibly one is used only once, and that is near the end. In the expectation of eventually finding a proof of the generalization of Theorem 2.3 which removes this assumption, throughout this proof we write $f(v)$ instead of 1. This also assists the reader in following the calculations, and adds no length to the proof. Note that to generalize Theorem 2.3, the concept of tree-connected would need to be developed considerably. This is discussed in Section 4.

Proof: Recall that the vertex set of H is $\{v_0, \ldots v_{r-1}\}$. For $v \neq v_0$, let $f(v) = 1$, and let $f(v_0) = \alpha > 1$. We shall construct H' with vertex set $\{v_0, \ldots v_{r-1}, v_r\}$ where for $v \neq v_0$, $f'(v) = 1$, and $f'(v_0) = \alpha - 1$. We shall show that H' is a tree-connected outline k-factorization. By repetition we obtain a tree-connected outline k-factorization with $f(v) = 1$ for each vertex v, which is in fact a connected k-factorization of K_{kn+1}.

H' is formed by adding the vertex v_r to H and replacing some of the loops on v_0 in H with edges from v_0 to v_r in H', and replacing some edges $\{v_z, v_0\}$, $1 \leq z \leq r-1$, in H with edges $\{v_z, v_r\}$ in H'.

To help us decide how to choose these loops and edges we create a bipartite graph B. It has vertex sets $\{v'_1, \ldots, v'_{r-1}\} \cup \{v(L)\}$ and $\{c'_1, \ldots, c'_n\}$. For $1 \leq z \leq r-1$, if v_z is joined to v_0 by $x_{i,z}$ edges of colour i in H, then in B there are $x_{i,z}$ edges joining v'_z to c'_i. Also $v(L)$ is joined to each c'_i by twice the number of loops of colour i on v_0 in H. So the degree of v'_z in B is equal to the total number of edges from v_z to v_0 in H, and the degree of $v(L)$ in B is equal to twice the number of loops on v_0 in H. The degree of c'_i in B is equal to the number of edges of colour i incident with v_0 in H (with loops counted twice). Thus

$$\begin{aligned} d_B(v'_z) &= \alpha f(v_z), & (2)\\ d_B(v(L)) &= \alpha(\alpha-1), \text{and} & (3)\\ d_B(c'_i) &= k\alpha, & (4) \end{aligned}$$

by $P(a)$, $P(b)$ and $P(c)$ respectively.

We need some more definitions. First note that we will write, for example, H_i and $C_{i,j}$ instead of $H_i(v_0)$ and $C_{i,j}(v_0)$. The following definitions also relate to v_0. Let $\omega_i^{(1)}$ be the number of components of H_i that are joined in $H(i)$ by just one edge to v_0, let $E_i^{(1)}$ be the set of edges in $H(i)$ from v_0 to these components, and let $\epsilon_i^{(1)} = |E_i^{(1)}|$. Note that $\omega_i^{(1)} = \epsilon_i^{(1)}$. Let $\omega_i^{(>1)}$ be the number of components of H_i joined by more than one edge to v_0 in $H(i)$, let $E_i^{(>1)}$ be the set of edges in $H(i)$ from v_0 to these components, and let $\epsilon_i^{(>1)} = |E_i^{(>1)}|$. Note that $\omega_i = \omega_i^{(1)} + \omega_i^{(>1)}$ and $\epsilon_i = \epsilon_i^{(1)} + \epsilon_i^{(>1)}$. Let $a_{i,j}$ and $b_{i,j}$ be defined by $\epsilon_{i,j} = a_{i,j}\alpha + b_{i,j}$ and $1 \leq b_{i,j} \leq \alpha$.

For $1 \leq i \leq n$, let S_i be the set of edges in B that correspond to loops on v_0 of colour i or edges of $E_i^{(>1)}$. Thus $|S_i| = \epsilon_i^{(>1)} + 2l_i$.

Claim 1 $|S_i| = \epsilon_i^{(>1)} + 2l_i \geq \alpha$.

Proof: As H is tree-connected it satisfies the tree bound $C(b)$. Thus

$$f(v_0) + \omega_i(v_0) \leq \epsilon_i(v_0) + l_i(v_0) + 1,$$

which we rewrite as

$$\alpha + \omega_i^{(1)} + \omega_i^{(>1)} \leq \epsilon_i^{(1)} + \epsilon_i^{(>1)} + l_i + 1.$$

Now, since $\omega_i^{(1)} = \epsilon_i^{(1)}$,

$$\alpha + \omega_i^{(>1)} - 1 \leq \epsilon_i^{(>1)} + l_i.$$

If $\omega_i^{(>1)} \geq 1$, then $\epsilon_i^{(>1)} + l_i \geq \alpha$ and the claim is proved. If $\omega_i^{(>1)} = 0$, then $\epsilon_i^{(>1)} = 0$ so we have

$$\alpha - 1 \leq l_i.$$

Hence

$$\begin{aligned} 2l_i &\geq 2\alpha - 2 \\ &= \alpha + (\alpha - 2) \\ &\geq \alpha, \end{aligned}$$

since $\alpha \geq 2$. $\square$

We form two laminar sets containing sets of edges of B. Let V^z be the set of edges incident with v'_z and let V^L be the set of edges incident with $v(L)$. Let $\mathcal{F}$ consist of V^z, $1 \leq z \leq r-1$, and V^L. $\mathcal{F}$ is a laminar set since the sets it contains are disjoint.

Let C^i be the set of edges incident with c'_i. Let $C^{i,j}$ be the set of edges of B that correspond to edges of H of colour i from v_0 to $C_{i,j}$. Note that

$$|C^{i,j}| = a_{i,j}\alpha + b_{i,j}. \tag{5}$$

Let $\mathcal{G}$ consist of C^i, $C^{i,j}$ and S_i, $1 \le j \le \omega_i$, $1 \le i \le n$. Note that for $1 \le i \le n$, $C^i \supseteq S_i$ and, for $1 \le j \le \omega_i$, $C^i \supseteq C^{i,j}$. If $\epsilon_{i,j} > 1$, then $S_i \supseteq C^{i,j}$; if $\epsilon_{i,j} = 1$, then $S_i \cap C^{i,j} = \emptyset$. Also $C^{i,j_1} \cap C^{i,j_2} = \emptyset$ if $j_1 \neq j_2$. Therefore $\mathcal{G}$ is a laminar set.

Apply Lemma 2.2 with M as the edge set of B and $h = \alpha$ to obtain a set of edges J that satisfies (1). Note that

$$\begin{aligned} |J \cap V^z| &= f(v_z), & (6)\\ |J \cap V(L)| &= \alpha - 1, & (7)\\ |J \cap C^i| &= k, & (8)\\ |J \cap C^{i,j}| &\le a_{i,j} + 1, & (9)\\ |J \cap S_i| &\ge 1, & (10) \end{aligned}$$

by (2), (3), (4), (5) and Claim 1 respectively.

We now form H'. We add a vertex v_r to H. If $\{v'_z, c'_i\} \in J$, then an edge coloured i from v_z to v_0 is replaced with an edge coloured i from v_z to v_r. If $\{v(L), c'_i\} \in J$, then a loop coloured i on v_0 in H is replaced with an edge coloured i from v_0 to v_r. We check that H' is a tree-connected outline k-factorization.

If $v, w \in V(H') \setminus \{v_0, v_r\}$, then the edges between v and w are unaffected. By (6), the number of edges from v_r to v_z is $f(v_z) = f'(v_z)f'(v_r)$. The number of edges from v_0 to v_z is, again by (6), reduced by $f(v_z)$ as H' is formed; so the number of such edges is now $\alpha f(v_z) - f(v_z) = (\alpha - 1)f(v_z) = f'(v_0)f'(v_z)$. The number of edges from v_0 to v_r is, by (7), $\alpha - 1 = f'(v_0)f'(v_r)$. So $P(a)$ is satisfied.

For $v \notin \{v_0, v_r\}$, the number of loops on v is unchanged. There are no loops on v_r (as required since $f'(v_r) = 1$). By (7), the number of loops on v_0 is reduced by $\alpha - 1$. The number that remain is $\binom{\alpha}{2} - (\alpha - 1) = \binom{\alpha-1}{2} = \binom{f'(v_0)}{2}$. So $P(b)$ is satisfied.

For $v \notin \{v_0, v_r\}$, v is incident with the same number of edges of each colour in H' as in H. By (8), v_r is incident with $k = kf'(v_r)$ edges of each colour. So in H' v_0 is incident with k fewer edges of each colour than it is in H. So the number that remain incident to v_0 in H' is $k\alpha - k = k(\alpha - 1) = kf'(v_0)$. So $P(c)$ is satisfied.

As $f'(v_0) + f'(v_r) = f(v_0)$, $P(d)$ is satisfied.

We have shown that H' is an outline k-factorization. It remains to show that it is tree-connected.

First we show that $C(a)$ is satisfied. To show that $H'(i)$ is connected it is sufficient to show that there is a path from v_0 to v_r since in H_i every vertex was in a component joined in $H(i)$ to v_0, so in H_i' every vertex is in a component joined in $H'(i)$ to either v_0 or v_r. We know by (10) that J contains at least one edge from S_i. If this is a loop, then v_0 is joined to v_r in $H'(i)$. The alternative is that J contains an edge from $E_i^{(>1)}$ that in $H(i)$ joins some $C_{i,j}$ to v_0. So this $C_{i,j}$ is joined to v_r in $H'(i)$. It remains joined to v_0 in H_i' if $C^{i,j} \setminus J \neq \emptyset$. We observe that

$$\begin{aligned} |C^{i,j} \setminus J| &= |C^{i,j}| - |C^{i,j} \cap J| \\ &\geq a_{i,j}\alpha + b_{i,j} - (a_{i,j} + 1), \end{aligned}$$

by (5) and (9). Therefore

$$\begin{aligned} |C^{i,j} \setminus J| &\geq a_{i,j}(\alpha - 1) + (b_{i,j} - 1) \\ &> 0, \end{aligned}$$

since if $a_{i,j} > 0$, then $a_{i,j}(\alpha - 1) > 0$ (as $\alpha \geq 2$), and if $a_{i,j} = 0$, then $b_{i,j} \geq 2$ and so $b_{i,j} - 1 > 0$.

We now show that $C(b)$ is satisfied. For $v \neq v_0$, we already know that $H'(i)$ is connected so $\epsilon_i'(v) \geq \omega_i'(v)$. Thus, as $f'(v) = 1$, $f'(v) + \omega_i'(v) \leq \epsilon_i'(v) + l_i'(v) + 1$. It only remains to show that $f'(v_0) + \omega_i'(v_0) \leq \epsilon_i'(v_0) + l_i'(v_0) + 1$; that is, $\alpha - 1 + \omega_i' \leq \epsilon_i' + l_i' + 1$.

Claim 2 $\alpha + \omega_i + \sum_{j=1}^{\omega_i} a_{i,j} \leq \epsilon_i + l_i + 1$.

Proof: If $\sum_{j=1}^{w_i} a_{i,j} = 0$ then the result follows from the fact that H satisfies $C(b)$. For other cases first recall that

$$\epsilon_i = \alpha \sum_j a_{i,j} + \sum_j b_{i,j}.$$

Since $\sum_j b_{i,j} \geq \omega_i$ we have

$$\epsilon_i \geq \alpha \sum_j a_{i,j} + \omega_i.$$

Therefore

$$\begin{aligned}
\epsilon_i + l_i + 1 &\geq \alpha \sum_j a_{i,j} + \omega_i + 1 \\
&= \alpha + \alpha \left(\sum_j a_{i,j} - 1 \right) + \omega_i + 1 \\
&= \alpha + (\alpha - 1) \left(\sum_j a_{i,j} - 1 \right) + \sum_j a_{i,j} - 1 + \omega_i + 1 \\
&= \alpha + \omega_i + \sum_j a_{i,j} + (\alpha - 1) \left(\sum_j a_{i,j} - 1 \right) \\
&\geq \alpha + \omega_i + \sum_j a_{i,j},
\end{aligned}$$

since $\alpha \geq 2$ and $\sum_j a_{i,j} \geq 1$. $\square$

Let $\Delta(\epsilon_i)$ be the number of edges (not loops) of colour i incident with v_0 in H that were replaced by edges of colour i incident with v_r in H'. Let $\Delta(l_i)$ be the number of loops of colour i on v_0 in H that were replaced by edges of colour i that join v_0 to v_r in H'. Thus $l'_i = l_i - \Delta(l_i)$ and $\epsilon'_i = \epsilon_i - \Delta(\epsilon_i) + \Delta(l_i)$. Hence

$$\epsilon'_i + l'_i = \epsilon_i + l_i - \Delta(\epsilon_i). \tag{11}$$

Now consider H'_i and H_i. H'_i contains the same components as H_i except that some (or perhaps none) of them are merged together and form a new component that also contains v_r. We want to know how many such components there are. That is, to how many different components of H_i are the $\Delta(\epsilon_i)$ edges moved from v_0 to v_r joined? Since, by (9), at most $a_{i,j} + 1$ of these edges are joined to $C_{i,j}$, $1 \leq j \leq \omega_i$, we can choose at most $\sum_j a_{i,j}$ of the $\Delta(\epsilon_i)$ edges to ignore in such a way that those left join v_0 to distinct components of H_i. So the number of components joined together to form a single component of H'_i that also contains v_r is at least $\Delta(\epsilon_i) - \sum_j a_{i,j}$. That is, H'_i has one component that contains vertices from at least $\Delta(\epsilon_i) - \sum_j a_{i,j}$ components of H_i. Therefore

$$\omega'_i - 1 \leq \omega_i - \left(\Delta(\epsilon_i) - \sum_j a_{i,j}\right).$$

Hence

$$\alpha - 1 + \omega'_i \leq \alpha + \omega_i - \Delta(\epsilon_i) + \sum_j a_{i,j}.$$

By Claim 2, we obtain

$$\alpha - 1 + \omega'_i \leq \epsilon_i + l_i - \Delta(\epsilon_i) + 1.$$

Then, by (11),

$$\alpha - 1 + \omega'_i \leq \epsilon'_i + l'_i + 1.$$

So $C(b)$ is satisfied and the Theorem is proved □

3 An Application

An interesting application of Theorem 2.3 is the following companion result to Theorem 1.2. Let K_t be edge-coloured on the set of colours $\{1, 2, \dots, n\}$. For $1 \leq i \leq n$, let $G(i)$ be the subgraph of K_t that is induced by the edges coloured i, and let $C_{i,1}, \dots, C_{i,\omega_i}$ be the components of $G(i)$. For $1 \leq i \leq n$ and $1 \leq j \leq \omega_i$, if $G(i)$ has maximum degree at most k then let $\epsilon_{i,j} = \sum_{v \in V(C_{i,j})}(k - d_{G(i)}(v))$ and let $\epsilon_i = \sum_{j=1}^{\omega_i} \epsilon_{i,j}$; so in particular, if $|V(C_{i,j})| = t$ then $2|E(G(i))| = kt - \epsilon_i$ for $1 \leq i \leq n$.

Theorem 3.1 *Let $k \geq 3$. Let $\alpha = kn + 1 - t$. An n-edge-coloured K_t, with colour classes $G(1), G(2), \dots, G(n)$ (where possibly some colour classes contain no edges), can be embedded in a n-edge-coloured K_{kn+1} in which each colour class is a connected k-factor if and only if*

(1) $d_{G(i)}(v) \leq k$ for each $v \in V(K_t)$ and for $1 \leq i \leq n$,

(2) $\epsilon_{i,j} \geq 1$ *for* $1 \leq i \leq n$ *and* $1 \leq j \leq \omega_i$,

(3) $\alpha \geq \max\{\epsilon_i/k : 1 \leq i \leq n\}$,

(4) $\alpha \geq \max\left\{\dfrac{2\omega_i - \epsilon_i - 2}{k-2} : 1 \leq i \leq n\right\}$, *and*

(5) if k *is odd then* n *is odd.*

Remark. If k is even, then Theorem 3.1 is equivalent to Theorem 1.2.

Proof: First suppose that K_t is embedded in a connected k-factorization of K_{kn+1}. Let $F(1), F(2), \ldots, F(n)$ be the k-factors where $G(i)$ is a subgraph of $F(i)$. We show that conditions (1-5) hold.

Since $d_{G(i)}(v) \leq d_{F(i)}(v) = k$ for each $v \in V(K_t)$ and for $1 \leq i \leq n$, we have that (1) holds.

For $1 \leq i \leq n$, since $F(i)$ is connected, each component of $G(i)$ must be connected to a vertex in $V(K_{kn+1}) \setminus V(K_t)$, so (2) holds.

Since, for $1 \leq i \leq n$, there are exactly ϵ_i edges in K_{kn+1} coloured i that join vertices in $V(K_t)$ to vertices in $V(K_{kn+1}) \setminus V(K_t)$, and since $d_{F(i)}(v) = k$ for each of the α vertices v in $V(K_{kn+1}) \setminus V(K_t)$, it must be that $\epsilon_i \leq k\alpha$, so (3) holds.

If the vertices in $V(K_{kn+1}) \setminus V(K_t)$ are amalgamated into a single vertex v_0, then by Lemma 2.1 the resulting graph H^* is a tree-connected outline k-factorization of K_{kn+1} with $f(v_0) = \alpha$ and $f(v) = 1$ for each $v \in V(H^*) \setminus \{v_0\}$. So H^* satisfies $C(b)$ for v_0:

$$\alpha + \omega_i \leq \epsilon_i + (k\alpha - \epsilon_i)/2 + 1.$$

Therefore $\alpha \geq (2\omega_i - \epsilon_i - 2)/(k-2)$, so (4) holds.

Finally, if k is odd then each k-factor is a graph in which each vertex has odd degree, so it must have an even number of vertices; so $kn + 1$ is even, and so (5) holds.

Now suppose that conditions (1-5) hold. We will use Theorem 2.4 to find the required embedding of K_t.

Let $V(K_t) = \{v_1, \ldots, v_t\}$. Form an n-edge-coloured graph H as follows. Add one vertex v_0 and

(a) for $1 \le i \le n$ and $1 \le z \le t$, join v_0 to v_z with $k - d_{G(i)}(v_z)$ edges coloured i, and

(b) for $1 \le i \le n$, place $(\alpha k - \epsilon_i)/2$ loops coloured i on v_0.

We will now check to see that the number of edges and loops added to form H are non-negative, so H is indeed a graph. Then we will show that H is a tree-connected outline k-factorization of K_{kn+1} with $f(v_0) = \alpha$ and $f(v_z) = 1$ for $1 \le z \le t$. The result will then follow from Theorem 2.3.

By (1), for $1 \le z \le t$ and $1 \le i \le n$ we know that $k - d_{G(i)}(v_z) \ge 0$, so by (a) it follows that $d_{H(i)}(v_z) = k = kf(v_z)$.

By (3), for $1 \le i \le n$ we know that $\alpha k \ge \epsilon_i$, and clearly

$$\begin{aligned}(\alpha k - \epsilon_i)/2 &= ((kn + 1 - t)k - \epsilon_i)/2 \\ &= k(kn+1)/2 - \epsilon_i - (kt - \epsilon_i)/2,\end{aligned}$$

which is an integer because $k(kn + 1)/2$ is an integer by (5) and because $(kt - \epsilon_i)/2 = |E(G(i))|$. Therefore $(\alpha k - \epsilon_i)/2$ is a non-negative integer. By (a) and (b) it follows that $d_{H(i)}(v_0) = (\epsilon_i + (\alpha k - \epsilon_i)) = \alpha k$.

We have now shown that H is a graph, and that it satisfies $P(c)$. For $1 \le z_1 < z_2 \le t$ clearly v_{z_1} and v_{z_2} are joined by $f(v_{z_1})f(v_{z_2}) = 1$ edge, and since for $1 \le z \le t$

$$\sum_{i=1}^{n}(k - d_{G(i)}(v_z)) = kn - (t-1) = \alpha,$$

v_0 is joined to v_z with $f(v_0)f(v_z) = \alpha$ edges, so H satisfies $P(a)$. Also, for $1 \le z \le t$, v_z is incident with $0 = \binom{f(v_z)}{1})$ loops, and v_0 is incident with $\sum_{i=1}^{n}(\alpha k - \epsilon_i)/2 = (\alpha kn - (ktn - t(t-1)))/2 = \alpha(\alpha - 1)/2 = \binom{f(v_0)}{2}$ loops (since $\sum_{i=1}^{n} \epsilon_i + \binom{t}{2} = \sum_{z=1}^{t} d_H(v_z) = kt$), so $P(b)$ is satisfied by H. Obviously $\sum_{z=0}^{t} f(v_z) = kn + 1$, so $P(d)$ is satisfied. Therefore H is an outline k-factorization of K_{kn+1}.

It remains to show that H is tree-connected. H satisfies $C(a)$ since by (2) for $1 \le i \le n$, each component of $G(i)$ contains a vertex that is joined by an edge coloured i to v_0. Also for $1 \le i \le n$, $\alpha \ge (2\omega_i - \epsilon_i - 2)/(k-2)$ by (4), so $k\alpha - \epsilon_i + 2\epsilon_i + 2 \ge 2\alpha + 2\omega_i$, so

$\alpha + \omega_i \leq \epsilon_i + l_i(v_0) + 1$, where $l_i(v_0) = (\alpha k - \epsilon_i)/2$ is the number of loops coloured i incident with v_0. Therefore $C(b)$ is satisfied when $v = v_0$, and is clearly satisfied for each other vertex v_z $(1 \leq z \leq t)$ since $f(v_z) = 1$ and since we have shown that $H(i)$ is connected. $\square$

4 Final Comments

In this paper we show that every amalgamation of a connected k-factorization of K_{kn+1} is a tree-connected outline k-factorization of K_{kn+1} (see Lemma 2.1). We then address the reverse question, asking which outline k-factorizations of K_{kn+1} are P-amalgamations of connected k-factorizations of K_{kn+1}. Theorem 2.3 answers this question in the case where P contains at most one element of size greater than one. This is a very useful result, allowing one to prove the embedding result in Theorem 3.1 which finds necessary and sufficient conditions for an n-edge-coloured K_{kn+1} in which each colour class is a connected k-factor.

It would be extremely interesting to find a generalization of Theorem 2.3 that removes the restriction that $f(v) = 1$ for all except possibly one vertex v. The example in Figure 1 shows that a stronger assumption on G than just being tree-connected will be needed. In the figure we show a $G(i)$ in a case with $k = 3$ and with $f(v) = 1$ for all vertices $v \in V(G(i))$ except for two, namely a and b, where $f(a) = f(b) = 2$. $G(i)$ is tree connected, but if b is split into two vertices of degree 3 then the resulting graph $G'(i)$ is not tree connected, so when a is split in its turn, the graph $G''(i)$ obtained cannot be connected. The stronger condition needed may well be provided by a theorem of Nash-Williams [15].

Nash-Williams solved the problem of deciding when a graph G with $V(G) = \{v_0, \ldots, v_{r-1}\}$ together with an associated function $f : V(G) \to \mathbb{N}$ is the $\{u_0, u_1, \ldots, u_{r-1}\}$-amalgamation of a single connected graph H in which $|u_i| = f(v_i)$ for $i \in \mathbb{Z}_r$. In a close approximation to Nash-Williams' terminology, we call H a $\{u_0, u_1, \ldots, u_{r-1}\}$-*detachment* of G. We would require that each colour class have a connected $\{u_0, u_1, \ldots, u_{r-1}\}$-detachment. (Of course, we would also require that the connected detachments form a partition of the edge-set of K_{kn+1}). Nash-Williams' result gave a necessary and sufficient condition for a graph to have a connected detachment satisfying some degree constraints; this condition reduces to the tree-bound in the case when $|u_i| = 1$ for $1 \leq i \leq r-1$. We make the following conjecture.

Conjecture. G is the amalgamation of a connected k-factorization of K_{kn+1} if and only if G is an outline k-factorization of K_{kn+1} in which, for $1 \leq i \leq n$, $G(i)$ has a k-regular connected detachment.

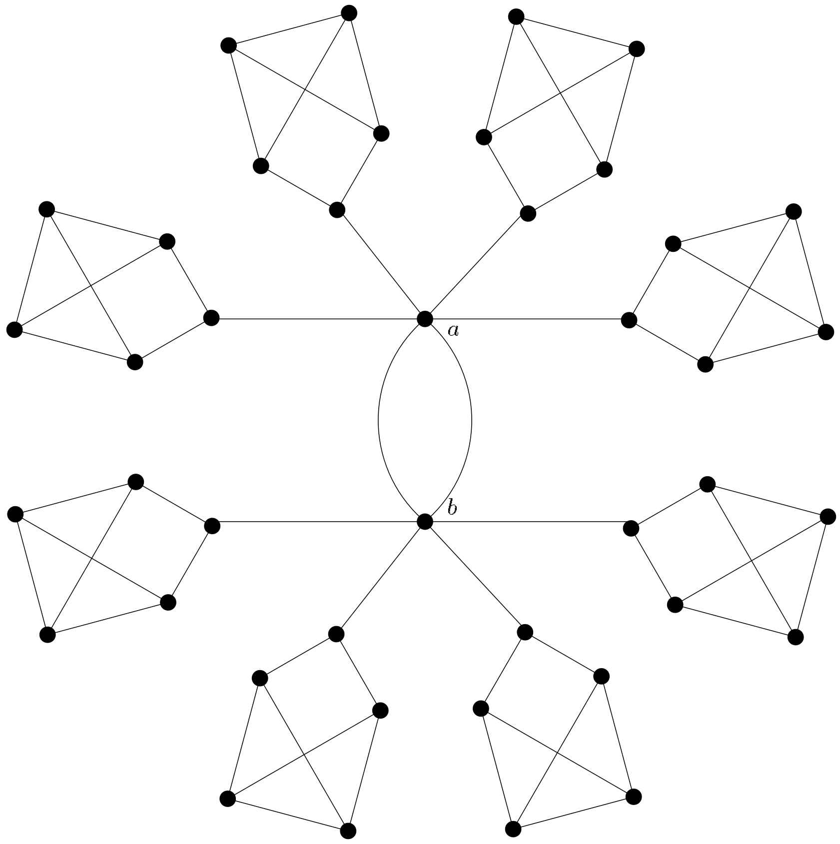

Figure 1.

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