Connectedness of the Graph of Vertex-Colourings

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Abstract

For a positive integer k and a graph G, the k-colour graph of G, $\mathcal{C}_k(G)$, is the graph that has the proper k-vertex-colourings of G as its vertex set, and two k-colourings are joined by an edge in $\mathcal{C}_k(G)$ if they differ in colour on just one vertex of G. In this note some results on the connectivity of $\mathcal{C}_k(G)$ are proved. In particular it is shown that if G has chromatic number $k \in \{2,3\}$, then $\mathcal{C}_k(G)$ is not connected. On the other hand, for $k \geq 4$ there are graphs with chromatic number k for which $\mathcal{C}_k(G)$ is not connected, and there are k-chromatic graphs for which $\mathcal{C}_k(G)$ is connected.

Keywords: vertex-colouring, k-colour graph, Glauber dynamics.

1 Introduction

Throughout this note a graph is finite, simple and loopless. Most of our terminology and notation will be standard and can be found in any textbook on graph theory such as, for example, [1]. For a positive integer k and a graph G, we define the *k-colour graph of G*, denoted $\mathcal{C}_k(G)$, as the graph that has the proper k-vertex-colourings of G as its vertex set, and two k-colourings are joined by an edge in $\mathcal{C}_k(G)$ if they differ in colour on just one vertex of G. In this note, we give some first results concerning the following question: given a graph G and a positive integer k, is $\mathcal{C}_k(G)$ connected?

This question has been looked at, in a certain sense, in the theoretical physics community when studying the Glauber dynamics of an anti-ferromagnetic Potts model at zero temperature. Associated with that research is the work on rapid mixing of Markov chains related to what we call the k-colour graph, in order to obtain efficient algorithms for almost uniform sampling of k-colourings of a graph. See, for instance, [3, 4] and references in those. But most

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of the work in those areas has concentrated on specific graphs such as finite parts of integer grids, or for values of k for which the connectedness of the k-colour graph was guaranteed. In this note we are interested in what can be said for general graphs and for small values of k.

A different colour graph, in which two k-colourings are adjacent if one can be obtained from the other by swapping the colours in a so-called *Kempe chain* (i.e., a connected component of the subgraph induced by the vertices coloured with one of two colours) has been considered in [6]. Note that our k-colour graph is a subgraph of this Kempe chain colour graph.

We will use $\alpha, \beta, \ldots$ to denote specific colourings. We say that G is *k-mixing* if $\mathcal{C}_k(G)$ is connected, and, having defined the colourings as vertices of $\mathcal{C}_k(G)$, the meaning of, for example, the path between two colourings should be clear. We assume throughout that $k \geq \chi(G)$ and that any k-colouring uses the colours $\{1, \ldots, k\}$.

If G has a k-colouring α, then we say that we can *recolour* G with β if $\alpha\beta$ is an edge of $\mathcal{C}_k(G)$; and if v is the unique vertex on which α and β differ, then we also say that we can *recolour* v. Given a k-colouring α, a colour is *available* for a vertex v if neither v nor any of its neighbours are assigned that colour.

In the next section we look for values of k that guarantee k-mixing; we obtain bounds in terms of the chromatic number, the maximum degree and the colouring number (also known as degeneracy or maximin degree). We also show that there exist graphs G for which k-mixing is not monotone, i.e., for which there exist numbers $k_1 < k_2$ so that G is k_2-mixing but not k_1-mixing.

In the two following sections we look at the case $k = \chi(G)$. It is shown that if $k = \chi(G)$ is 2 or 3, then G is not k-mixing. On the other hand, for all $k \geq 4$ there are graphs with chromatic number k that are not k-mixing and graphs with chromatic number k that are k-mixing.

The results from the earlier sections make it possible to characterise all positive integers L and sets P with $\min P \geq L$ such that there exist graphs G with $\chi(G) = L$ that are k-mixing if and only if $k \notin P$. This result can be found in the final section.

2 First results on mixing

One might expect that if k is sufficiently large compared to the chromatic number of a graph, then the graph will be k-mixing. We first show that no such result is possible.

For $m \geq 3$, let L_m be the graph obtained from the balanced complete bipartite graph $K_{m,m}$ by removing the edges of a perfect matching in $K_{m,m}$. Note that L_m is bipartite, and hence has chromatic number 2. It is also obvious that there are many ways to colour L_m with m colours. But suppose that we colour the vertices in each part of the bipartition of L_m with the colours $1, 2, \ldots, m$, where vertices in opposite parts that were originally connected by an edge from the removed perfect matching are given the same colour. It is easy to check that this m-colouring is an isolated node in the k-colour graph $\mathcal{C}_m(L_m)$. Hence L_m is not m-mixing, proving the following.

Property 1
There is no expression $\varphi(\chi)$ in terms of the chromatic number χ, so that for all graphs G and integers $k \geq \varphi(\chi(G))$, G is k-mixing.

From now on we will use the term *frozen* for a k-colouring of a graph G that forms an isolated node in the k-colour graph. For $k \geq 2$, the existence of frozen k-colourings of a graph will immediately imply that the graph is not k-mixing.

The graphs L_m have more interesting properties : they are k-mixing for all $3 \leq k \leq m-1$. To see this, consider a k-colouring of L_m with $3 \leq k \leq m-1$, and suppose L_m has bipartition $\{X, Y\}$. Since X contains m vertices, there is at least one colour c_1 that appears on more than one vertex of X. But that means that no vertex in Y has been coloured with c_1. Hence it is possible to recolour all vertices in X with this colour c_1. Once that is done, we can choose a second colour $c_2 \neq c_1$ and recolour every vertex in Y with c_2. This way we have shown that any k-colouring of L_m is connected to some 2-colouring of L_m. It is an easy exercise to show that if $k \geq 3$, all 2-colourings of L_m are connected in $\mathcal{C}_k(L_m)$, thus showing that $\mathcal{C}_k(L_m)$ is connected for $3 \leq k \leq m-1$.

If we colour L_m with $k \geq m+1$ colours, then we again have that a certain colour is not used on Y. So, by a similar argument to the case above, it follows that $\mathcal{C}_k(L_m)$ is connected for $k \geq m+1$. We summarise the properties of the graphs L_m.

Property 2
For $m \geq 3$, the graph L_m is a bipartite graph that is k-mixing for $3 \leq k \leq m-1$ and $k \geq m+1$, but not k-mixing for $k = m$.

Recall that the *colouring number* $\mathrm{col}(G)$ of a graph G (which is also known as the *degeneracy* or the *maximin degree*) is defined as the largest minimum degree of any subgraph of G. That is, $\mathrm{col}(G) = \max\limits_{H \subseteq G} \delta(H)$. The following result is stated in [2] with the lower bound one larger, although the proof in [2] is essentially the proof we give below.

Theorem 3
For any graph G and integer $k \geq \mathrm{col}(G)+2$, $\mathcal{C}_k(G)$ is connected.

Proof : We use induction on the number of vertices of G. The result is obviously true for the graph with one vertex. So suppose G has two or more vertices. Let v be a vertex with degree $d_G(v) \leq \mathrm{col}(G)$, and set $G' = G - \{v\}$. Note that $\mathrm{col}(G') \leq \mathrm{col}(G)$, hence we also have $k \geq \mathrm{col}(G')+2$. By induction we can assume that $\mathcal{C}_k(G')$ is connected.

Take two k-colourings α and β of G, and let α', β' be the k-colourings of G' induced by α, β. Since $\mathcal{C}_k(G')$ is connected, there exists a sequence $\alpha' = \gamma'_0, \gamma'_1, \ldots, \gamma'_N = \beta'$ of k-colourings of G' so that for $i = 1, \ldots, N$, γ'_{i-1} and γ'_i differ in the colour of exactly one vertex of G'. Denote this vertex by v_i and denote the new colour $\gamma'_i(v_i)$ by c_i. We now try to take the same recolouring steps to recolour G, starting from α. If for some i it is not possible to recolour vertex v_i, this must be because v_i is adjacent to v and v at that moment has the colour c_i. But because v has degree at most $\mathrm{col}(G) \leq k-2$, there is a colour $c \neq c_i$ that does not appear on any of the neighbours of v. Hence we can first recolour v to c, and then continue with recolouring v_i to c_i and move on.

In this way we find a sequence of k-colourings of G, starting at α, and ending in a colouring in which all the vertices except possibly v will have the same colour as in β. But then, if necessary, we can also recolour v to give it the colour from β. This gives a path between α and β in $\mathcal{C}_k(G)$, completing the proof. $\square$

Since the maximum degree $\Delta(G)$ of a graph G is at most the colouring number $\mathrm{col}(G)$, Theorem 3 immediately means that for $k \geq \Delta(G)+2$, $\mathcal{C}_k(G)$ is connected. It is believed that the Glauber dynamics Markov chain is rapidly mixing for $\Delta(G)+2$ or more colours, [3]. The best known lower bound on the number of colours needed for rapid mixing is $\frac{11}{5}\,\Delta(G)$, [7].

Note that the expressions in terms of the colouring number cannot guarantee rapid mixing of the Glauber dynamics Markov chain. For instance, the stars $K_{1,m}$ have colouring number $\mathrm{col}(K_{1,m}) = 1$. But it is shown in [5] that the Glauber dynamics Markov chain for those graphs is not rapidly mixing for $k \leq m^{1-\varepsilon}$, for fixed $\varepsilon > 0$.

There are many graphs that show the bound in Theorem 3 is best possible. For instance the graphs L_m defined at the beginning of this section have $\mathrm{col}(L_m) = m-1$ and are not m-mixing. Even simpler, the complete graphs K_n have $\mathrm{col}(K_n) = n-1$, but are not n-mixing since every n-colouring of a complete graph is a frozen colouring.

3 Graphs with chromatic number 2 or 3

We briefly consider the case of graphs with chromatic number 2 — that is, bipartite graphs with at least one edge. A graph that has chromatic number 2 and is connected has just two frozen 2-colourings. In general, if $\chi(G) = 2$, then there is a path between a pair of 2-colourings of G if and only if they agree on every connected component that contains more than one vertex. It is an easy exercise to show that if G is a bipartite graph with p isolated vertices and q other connected components, then $\mathcal{C}_2(G)$ has 2^q connected components, each of which is a p-dimensional cube.

In the remainder of this section, we consider graphs with chromatic number 3. We first present Lemma 4 that describes how we might be able to recognise that two 3-colourings of a graph are not connected by looking only at the colours of vertices that lie on a cycle. We use this to prove that 3-chromatic graphs are not 3-mixing.

To orient a cycle means to orient each edge on the cycle so that a directed cycle is obtained. If C is a cycle, then by $\overrightarrow{C}$ we denote the cycle with one of the two possible orientations. Given a 3-colouring α, the weight of an edge $e = uv$ oriented from u to v is

$$w(\overrightarrow{uv}, \alpha) \;=\; \begin{cases} +1, & \text{if } \alpha(u)\alpha(v) \in \{12,\ 23,\ 31\}; \\ -1, & \text{if } \alpha(u)\alpha(v) \in \{21,\ 32,\ 13\}. \end{cases}$$

The weight $W(\overrightarrow{C}, \alpha)$ of an oriented cycle $\overrightarrow{C}$ is the sum of the weights of its oriented edges.

Lemma 4 *Let α and β be 3-colourings of a graph G that contains a cycle C. Then if α and β are in the same component of $\mathcal{C}_3(G)$, we must have $W(\overrightarrow{C}, \alpha) = W(\overrightarrow{C}, \beta)$.*

We note that the converse is not true. Given a 3-colouring of an oriented 3-cycle, obtain a second colouring by changing the colour on each vertex to that of its unique out-neighbour in the original colouring. The two colourings are not connected — they are both frozen — but the weight of the cycle is the same for each.

Proof of Lemma 4: Let α and α' be 3-colourings of G that are adjacent in $\mathcal{C}_3(G)$. And suppose the two 3-colourings differ on vertex v. If v is not on C, then we certainly have $W(\overrightarrow{C}, \alpha) = W(\overrightarrow{C}, \alpha')$.

If v is a vertex of C, then its two neighbours on C must have the same colour in α (otherwise we wouldn't be able to recolour v). If we denote the in-neighbour of v on $\overrightarrow{C}$ by v_i and its out-neighbour by v_o, then this means that $w(\overrightarrow{v_iv},\alpha)$ and $w(\overrightarrow{vv_o},\alpha)$ have opposite sign, hence $w(\overrightarrow{v_iv},\alpha)+w(\overrightarrow{vv_o},\alpha)=0$. Recolouring vertex v will change the signs of the weights of the oriented edges $\overrightarrow{v_iv}$ and $\overrightarrow{vv_o}$, but they will remain opposite. Therefore $w(\overrightarrow{v_iv},\alpha')+w(\overrightarrow{vv_o},\alpha')=0$, and it follows that $W(\overrightarrow{C},\alpha)=W(\overrightarrow{C},\alpha')$.

From the above we immediately obtain that the weight of an oriented cycle is constant on all 3-colourings in the same component of $\mathcal{C}_3(G)$ □

Lemma 5 *Let α be a 3-colouring of a graph G that contains a cycle C. If $W(\overrightarrow{C},\alpha)\neq 0$, then $\mathcal{C}_3(G)$ is not connected.*

Proof : Let β be the 3-colouring of G obtained by setting for each vertex v of G :

$$\beta(v) \;=\; \begin{cases} 1, & \text{if } \alpha(v)=2; \\ 2, & \text{if } \alpha(v)=1; \\ 3, & \text{if } \alpha(v)=3. \end{cases}$$

It is easy to check that for each edge e in C, $w(\vec{e},\alpha)=-w(\vec{e},\beta)$, which gives $W(\overrightarrow{C},\alpha)=-W(\overrightarrow{C},\beta)$. Since $W(\overrightarrow{C},\alpha)\neq 0$, we must have $W(\overrightarrow{C},\alpha)\neq W(\overrightarrow{C},\beta)$, and so, by Lemma 4, α and β belong to different components of $\mathcal{C}_3(G)$. □

Theorem 6
Let G be a graph with chromatic number 3. Then $\mathcal{C}_3(G)$ is not connected.

Proof : As G has chromatic number 3, it is not bipartite and hence contains a cycle C of odd length. Let α be a 3-colouring of G, and note that as the weight of each edge in $\overrightarrow{C}$ is $+1$ or -1, $W(\overrightarrow{C},\alpha)\neq 0$. We are done by Lemma 5. □

For an even cycle C_{2m} with $2\,m\geq 6$, it is easy to construct a 3-colouring α of C_{2m} so that $W(\overrightarrow{C}_{2m},\alpha)\neq 0$. (Use the colour pattern $1,2,3,1,2,3,\ldots$ as long as possible — making sure that the final vertices are properly coloured.) By Lemma 5 we can conclude that for all even $2\,m\geq 6$, the cycle C_{2m} is not 3-mixing.

We leave it to the reader to check that the 4-cycle C_4 is the only cycle that is 3-mixing.

4 Graphs with chromatic number at least 4

For any $k\geq 4$, it is easy to find graphs with chromatic number k that are not k-mixing; for example, K_k or any k-chromatic graph that contains it as an induced subgraph. In this section, we show that, in contrast to the results of the previous section on graphs with chromatic number 2 or 3, for $k\geq 4$, there exist graphs with chromatic number k that are k-mixing.

For $m\geq 4$, the graph H_m is defined as follows : the vertex set is $\{u,v_1,v_2,\ldots,v_{m-1},w_1,w_2,\ldots,w_{m-1}\}$, and

- for $1\leq i<j\leq m-1$, there are edges v_iv_j and w_iw_j;
- for $2\leq i\leq m-1$, there are edges uv_i and uw_i; and
- there is an edge v_1w_1.

We remark that H_m is obtained from two copies of K_m using Hajos' construction; see, for example, [1]. This implies that H_m has chromatic number m and, moreover, that it is m-critical. (Removing any vertex or edge from H_m will lead to a graph with chromatic number less than m.)

In this section we will prove the following properties of H_m.

Property 7
For $m \geq 4$, the graph H_m is an m-chromatic graph that is k-mixing for all $k \geq m$.

The fact that H_m is k-mixing for $k \geq m+1$ follows immediately from Theorem 3. We shall show that H_m is m-mixing as well.

We divide the m-colourings of H_m into classes according to the colour of v_1 and w_1. An m-colouring α is a (c, c')-*colouring* if $\alpha(v_1) = c$ and $\alpha(w_1) = c'$. If $\alpha(u) = c$ also, we call α a *standard* (c, c')-colouring.

We will show that H_m is m-mixing by showing that

- every m-colouring is connected to a standard colouring;
- for any pair c, c', the set of all standard (c, c')-colourings is connected; and
- for any two pairs c, c' and d, d', each standard (c, c')-colouring is connected to a standard (d, d')-colouring.

Lemma 8 *Let c and c' be distinct colours. Let α be a (c, c')-colouring of H_m where $\alpha(u) = c''$. Then there is a path from α to a standard (c, c')-colouring or to a standard (c'', c')-colouring of H_m.*

Proof: We assume $c \neq c''$ (else there is nothing to prove). Note that as $\alpha(v_1) = c$, $\alpha(v_i) \neq c$ for $2 \leq i \leq m-1$. If it is not possible to immediately recolour u with c to obtain a standard (c, c')-colouring, then there must be a vertex w_j, $j \in \{2, \ldots, m-1\}$, such that $\alpha(w_j) = c$.

If $c'' = c'$, then, as two of the $m-1$ neighbours of w_j are coloured c', there is some colour d not used on either w_j or any of its neighbours. Recolour w_j with d and then u with c to obtain a standard (c, c')-colouring.

If $c'' \neq c'$, then no neighbour of v_1 is coloured c''. By recolouring v_1 with c'', we immediately obtain a standard (c'', c')-colouring. □

Lemma 9 *For each distinct pair of colours c and c', all standard (c, c')-colourings belong to the same connected component of $\mathcal{C}_m(H_m)$.*

Proof: Let α and β be distinct standard (c, c')-colourings and let x be the first vertex in the ordering $v_2, \ldots, v_{m-1}, w_2, \ldots, w_{m-1}$ at which α and β disagree. To prove the lemma, we show that from α we can recolour to obtain a colouring that agrees with β on x and all vertices prior to it in the ordering.

Suppose that $x = v_i$ for some $i \in \{2, \ldots, m-1\}$. We simply recolour v_i with $\beta(v_i)$ unless there is a vertex v_j such that $\alpha(v_j) = \beta(v_i)$; in which case, by the choice of x, $j > i$. Note that a total of $m-1$ colours are used on $u, v_1, \ldots, v_{m-1}$ in any standard (c, c')-colouring, so there is a colour d available for v_j. Recolour v_j with d and then recolour v_i with $\beta(v_i)$.

The other possibility is that $x = w_i$ for some $i \in \{2, \ldots, m-1\}$. Much as before, recolour w_i with $\beta(w_i)$ unless there is a vertex w_j, $j > i$, such that $\alpha(w_j) = \beta(w_i)$. If there is a colour d available at w_j, then recolour w_j with d and then recolour w_i with $\beta(w_i)$. In this case, however, there is not necessarily a colour available at w_j. If there is not, find, if necessary, a vertex $v_l \in \{v_2, \ldots, v_{m-1}\}$ coloured c' and recolour it with its available colour. In any case, u can now be recoloured c' and so c is now available at w_j. Finally we perform the following sequence of recolourings: w_j with c, w_i with $\beta(w_i)$, w_j with $\alpha(w_i)$, u with c and, if such a vertex was found, v_l with $\alpha(v_l)$. □

Lemma 10 *Let α be a standard (c, c')-colouring of H_m. Then there is a path from α to a standard (c', c'')-colouring of H_m for any $c'' \notin \{c, c'\}$.*

Proof: From α, we describe a sequence of recolourings that lead to a standard (c', c'')-colouring. First, if one of $v_2, \ldots, v_{m-1}$ is coloured c', it is recoloured with its available colour. Then u is recoloured c'. Next, if one of $w_2, \ldots, w_{m-1}$ is coloured c'', it is recoloured c. Then w_1 is recoloured c'' and v_1 is recoloured c'. □

Lemma 11 *For each $m \geq 4$, H_m is m-mixing.*

Proof: Let α and β be two m-colourings of H_m; we must show that they are connected. By Lemma 8, we can assume that they are standard colourings. So suppose that α is a standard (c, c')-colouring and that β is a standard (d, d')-colouring. By Lemma 9, it is sufficient to find a path from α to any standard (d, d')-colouring. There are a number of cases.

Suppose that $d = c'$. If $d' \neq c$, then the theorem follows immediately from Lemma 10. If $d' = c$, then, let b and b' be distinct colours not in $\{c, c'\}$. (As $m \geq 4$, such colours can be found. This need to have four colours available, explains, in essence, why the theorem is not correct for smaller m.) Now we repeatedly apply Lemma 10: from α we can find a path to a standard (c', b)-colouring, then to a standard (b, b')-colouring, then a standard (b', c')-colouring and finally a standard (c', c)-colouring.

Suppose that $d = c$. Then if $d' = c'$ the result follows from Lemma 9. Otherwise, applying Lemma 10, we find a path from α to a standard (c', b)-colouring (for some distinct colour b), then to a standard (b, c)-colouring, and then to the required standard (c, d')-colouring.

If $d \notin \{c, c'\}$, then Lemma 10 gives a path from α to a standard (c', d)-colouring and then to a standard (d, d')-colouring. □

5 Graphs that are mixing only for permitted values

In this section we use some results from the previous sections to prove the following.

Theorem 12
Let $L \geq 2$ be an integer, and P a set of integers, with $\min P \geq L$ if $P \neq \varnothing$. Then the following two statements are equivalent:
(a) There exists a graph G with chromatic number L such that for all $k \geq L$, G is k-mixing if and only if $k \notin P$.
(b) The set P is finite, and if $L \in \{2, 3\}$, then $L \in P$.

By Theorem 3, a graph can be non-k-mixing for a finite number of k only. Also, by the results of Section 3, a graph with chromatic number $L \in \{2,3\}$ cannot be L-mixing. Hence statement (a) implies (b).

Recall the graphs from Sections 2 and 4 :

- for $m \geq 3$, L_m has chromatic number 2 and is k-mixing if and only if $k \geq 3$ and $k \neq m$;
- for $m \geq 4$, H_m has chromatic number m and is k-mixing if and only if $k \geq m$.

We also have the trivial observation :

- for $m \geq 2$, the complete graph K_m has chromatic number m and is k-mixing if and only if $k \geq m+1$;

If a graph G is the disjoint union of graphs $G_1, \ldots, G_s$, then we obviously have that $\chi(G)$ is $\max\{\, \chi(G_i) \,:\, i=1,\ldots,s \,\}$, and G is k-mixing if and only each G_i, $1 \leq i \leq s$, is k-mixing.

Now let L and P be as in the theorem and suppose statement (b) holds. If $P = \varnothing$, we are in the case $L \geq 4$ and the graph H_L will do the trick for (a).

So we can assume that P is not empty but finite. Write $P = \{p_1, \ldots, p_t\}$ with $p_1 = \min P$. Then if $L \in P$, hence $p_1 = L$, the disjoint union of $K_L, L_{p_2}, \ldots, L_{p_t}$ has chromatic number L, and for $k \geq L$, the graph is k-mixing if and only if $k \notin P$. Finally, if $L \notin P$, we must have $p_1 > L \geq 4$, and then the disjoint union of $H_L, L_{p_1}, \ldots, L_{p_t}$ will provide a graph for which (a) holds.

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