Characterization of graphs with Hall number 2

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Abstract

Hall's condition is a simple condition that a graph G and list assignment L must satisfy if G has a proper L-colouring. The Hall number of G is the smallest integer m such that whenever each list has size m and Hall's condition is satisfied a proper L-colouring exists. Hilton and P.D. Johnson introduced the parameter and showed that a graph has Hall number 1 if and only if every block is a clique. In this paper we characterize graphs with Hall number 2.

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1 Introduction

In this paper G will denote a finite simple connected graph and L will denote a *list assignment* to the vertices of G, that is, a function from $V(G)$ to the collection of finite subsets of a set (of *colours*), C. A *proper L-colouring* of G is a selection $\phi(v) \in L(v)$ for all $v \in V(G)$ such that if u and v are adjacent vertices in G, then $\phi(u) \neq \phi(v)$. In other words, $\phi\colon V(G) \longrightarrow C$ is a proper L-colouring if for each $\sigma \in C$, $\phi^{-1}(\sigma) = \{v \in V(G) : \phi(v) = \sigma\}$ is an independent set of vertices in G.

Let $\alpha(\sigma, L, H)$ be the independence number of the subgraph of a graph H induced by the set of vertices $\{u \in V(H) : \sigma \in L(u)\}$. Then $\alpha(\sigma, L, H)$ is not less than the largest number of vertices that can be coloured with σ in a proper L-colouring of H. Thus if H has a proper L-colouring we must have

$$|V(H)| \leq \sum_{\sigma \in C} \alpha(\sigma, L, H). \tag{1}$$

We say that G and L satisfy *Hall's condition* if and only if (1) is satisfied for every subgraph H of G. Hall's condition is obviously necessary for the existence of a proper L-colouring of G. In fact, for G and L to satisfy Hall's condition it can be seen that it suffices that (1) holds for all induced subgraphs of G. The reason for the name of the condition is that in the case where G is a clique, a proper L-colouring of G is also a system of distinct representatives (SDR) of the sets $L(v)$, $v \in V(G)$, and in this case Hall's condition is equivalent to the necessary and sufficient condition for the existence of an SDR given in the famous theorem of Phillip Hall [3].

The *Hall number* of a graph G, denoted $h(G)$, is the smallest integer m such that there is a proper L-colouring of G whenever $|L(v)| \geq m$ for all $v \in V(G)$ and Hall's condition is satisfied. The problem of which graphs have Hall number 1 was settled by Hilton and P. D. Johnson.

Theorem 1 [4] *A graph G has Hall number 1 if and only if every block of G is a clique.*

The main result of this paper is a characterization of graphs with Hall number 2. This result is the culmination of the work of a number of authors, particularly Eslahchi, Hilton, P. D. Johnson and Wantland [2, 4, 7], on whose results we rely greatly. We observe in passing that Eslahchi has characterized the line graphs with Hall number 2 [1].

Why do we study Hall numbers? There are two reasons. First, as the name of the numbers suggests, we are interested in discovering more on systems of distinct representatives, and looking at how generalizations of Hall's theorem can be developed. Second, the reason for looking at Hall numbers in terms of graph colourings is their relation to the choice and chromatic numbers. The choice number $c(G)$ is the smallest integer m such that there is a proper L-colouring of G whenever $|L(v)| \geq m$ for all $v \in V(G)$. It is clear that $c(G) \geq \chi(G)$, the chromatic number of G, but determining for which graphs G, $c(G) = \chi(G)$ is an open problem. The relation of Hall numbers to this problem is detailed in [5, 6]. One result is that $c(G) = \chi(G)$ if and only if $h(G) \leq \chi(G)$. So the problem of characterizing graphs G such that $h(G) \leq k$ for any integer k is of definite interest in the quest for solutions to the equation $c(G) = \chi(G)$. In particular, $h(G) \leq 2$ implies that $c(G) = \chi(G)$.

2 Results

First we discuss definitions and notation. Let $m_1, \ldots, m_k$ be positive integers, at most one of which is 1. Then $\theta(m_1, \ldots, m_k)$ is the graph constructed by joining two vertices by k internally disjoint paths of lengths $m_1, \ldots, m_k$. Let $k \geq 2$ and $a_1, \ldots, a_k$ be positive integers such that $m = \sum_{i=1}^{k} a_i \geq 3$. Then the *partial wheel graph* $W(a_1, \ldots, a_k)$ is the graph that contains the cycle C_m plus one other vertex that is adjacent to k of the vertices in the cycle where the paths around the cycle between vertices of degree 3 are, in one orientation of the cycle, of lengths $a_1, \ldots, a_k$. The edges not in the cycle are called *radial* edges. Let $r_1, \ldots, r_k$ be positive integers. Then the *partial wheel-like graph* $WL(r_1, \ldots r_k; a_1, \ldots, a_k)$ is obtained from $W(a_1, \ldots, a_k)$ by replacing the radial edge that is incident with the vertex that lies between arcs of lengths a_{j-1} and a_j (subtraction is mod$k$) by a path of length r_j. Examples of a partial wheel graph and a partial wheel-like graph are dis-

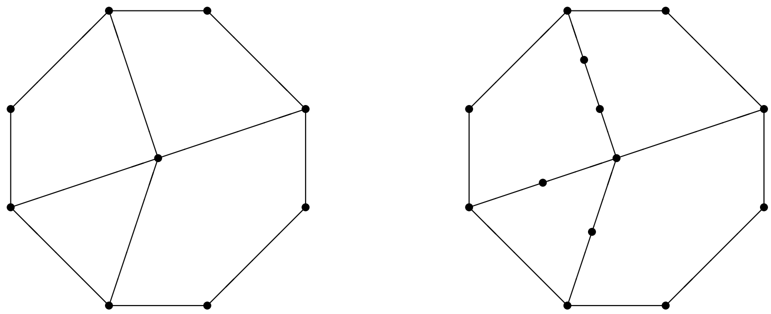


Figure 1: $W(2,3,1,2)$ and $WL(3,1,2,2;2,3,1,2)$

played in Figure 2.

If a path of length m is added to a graph G between two vertices u and v of G, then the graph obtained is called G *with an ear of length* m *(attached to* u *and* v*).* If $m = 2$ and u and v are adjacent in G, then the new graph can also be called G *with a triangle based on* uv. Let G and H be graphs, let l be a nonnegative integer. Then *cuff* (G, H, l) is the graph obtained by joining G and H by a path of length l; sometimes it is necessary to specify the vertices at which the path is attached to G and H.

The *core* of a graph is the subgraph obtained by successively removing vertices of degree 1 until none remain. A *block* of a graph is a maximal connected subgraph with no cutvertices.

We present the main theorem of this paper.

Theorem 2 *The core of a connected graph has Hall number 2 if and only if the core is one of the following graphs:*

1. C_n, $n \geq 4$,
2. $\theta(m, 2, 1)$, $m \geq 2$,
3. $\theta(m, 2, 2)$, $m \geq 2$,
4. $\theta(3, 3, 2)$,
5. K_4 *with an ear of length 2,*

6. *any cycle of even length with two triangles based on non-adjacent edges of the cycle, or*

7. *a graph G with at least two blocks in which every block is either a clique, $\theta(2,2,1)$ or K_4 with an ear of length 2; at least one block is not a clique; and if a block is $\theta(2,2,1)$ or K_4 with an ear, then the only vertices in the block that can be cutvertices in G are the vertices of degree 2.*

Theorem 3 *A graph has Hall number 2 if and only if its core has Hall number 2.*

The combination of Theorems 2 and 3 gives a characterization of graphs with Hall number 2.

Proof of Theorem 3: Sufficiency was proved in [7, Corollary 2]. If the core of a graph has Hall number 1, then the graph has Hall number 1 (Theorem 1). It is easy to show (and was proved in [6]) that h is monotone with respect to taking induced subgraphs; that is, if H is an induced subgraph of G, then $h(H) \leq h(G)$. Therefore if the core of a graph has Hall number greater than 2, then the graph has Hall number greater than 2. $\square$

Notice that we can divide the graphs in Theorem 2 into those with one block (1-6), and those with two or more blocks (7) (a graph with no blocks in its core has Hall number 1). The graphs with one block have previously been shown to have Hall number 2.

Theorem 4 [2, 6] *The following graphs have Hall number 2:*

1. C_n, $n \geq 4$,

2. $\theta(m,2,1)$, $m \geq 2$,

3. $\theta(m,2,2)$, $m \geq 2$,

4. $\theta(3,3,2)$,

5. *K_4 with an ear of length 2,*

6. *any cycle of even length with two triangles based on non-adjacent edges of the cycle.*

In the next section we shall show that there is no other core of a graph with one block with Hall number 2. In the final section we complete the proof of Theorem 2 by considering graphs with more than one block in their core.

We need to know how to show that a graph has Hall number greater than 2. As h is monotone with respect to taking induced subgraphs, for each positive integer k, the collection of graphs with Hall number at most k has a "forbidden-induced-subgraph" characterization: if we define a graph H to be *Hall-k^+-critical* if and only if $h(H) > k$ but $h(H - v) \leq k$ for every $v \in V(H)$, then $h(G) \leq k$ if and only if G has no Hall-k^+-critical induced subgraph. Thus the quest for a characterization of graphs with Hall number 2 has centred on a search for Hall-2^+-critical graphs; many have been found.

Theorem 5 [2, 6, 7] *The following graphs are Hall-2^+-critical:*

1. *$\theta(m_1, m_2, m_3)$, $m_1 \geq m_2 \geq m_3$, $m_2 \geq 3$ except when $(m_1, m_2, m_3) = (3, 3, 2)$,*

2. *$\theta(m, 2, 2, 1)$ and $\theta(m, 2, 2, 2)$, $m \geq 2$, and $\theta(3, 3, 2, 2)$.*

3. *K_5 with an ear of length 2,*

4. *K_4 with an ear of length m, $m \geq 3$,*

5. *K_4 with two disjoint ears each of length 2,*

6. *two K_4's intersecting in a K_2,*

7. *K_5 minus an edge,*

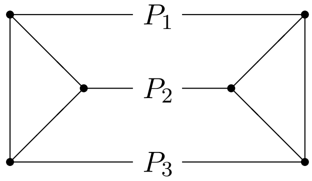
8. *any cycle with two triangles based on adjacent edges of the cycle,*

9. *any cycle with two triangles based on the same edge of the cycle,*

10. *any cycle of odd length with two triangles based on non-adjacent edges of the cycle,*

11. *any cycle of even length with three triangles based on non-adjacent edges of the cycle,*

12. $G_1 =$



where P_1*,* P_2 *and* P_3 *are paths such that either two of them are single edges or the lengths of all three have the same parity,*

13. $W(a_1, a_2, 1)$*, for any* (a_1, a_2) *except* $(1, 1)$*,*

14. $W(1, 1, 1, 1)$*,*

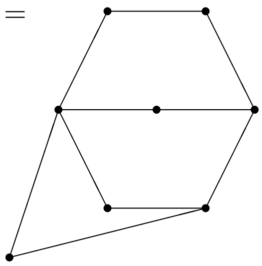
15. $WL(r_1, r_2, r_3; 1, 1, 1)$*,* $r_1 \geq 2$*,*

16. $WL(r_1, 1, 1; a_1, 1, a_3)$*,* $r_1 \geq 2$*,* $a_1 + a_3 \geq 3$*,*

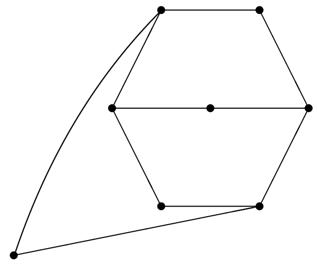
17. $WL(r, 1, 1; 1, a, 1)$*,* $r \geq 2$*,* $a \geq 2$*,*

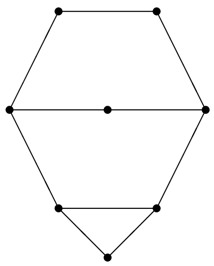
18. $WL(2, 2, 1; 1, 2, 2)$*,*

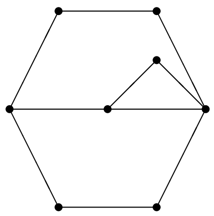
19. $G_2 =$



20. $G_3 =$



21. $G_4 =$ 

22. $G_5 =$ 

23. *Cuff* (C_{m_1}, C_{m_2}, l), $m_1 \geq 4$, $m_2 \geq 3$, $l \geq 0$

24. *Cuff* $(\theta(2,2,1), K_3, l)$, $l \geq 0$, *provided the point of attachment of the joining path to* $\theta(2,2,1)$ *is one of the vertices of degree 3,*

25. *Cuff* $(\theta(m,2,1), K_3, l)$, $m \geq 3$, $l \geq 0$, *provided the point of attachment of the joining path to* $\theta(m,2,1)$ *is the vertex of degree 2 in* $\theta(m,2,1)$ *that is adjacent to the 2 vertices of degree 3,*

26. *Cuff* $(K_4$ *with an ear of length 2,* $K_3, l)$, $l \geq 0$, *provided the point of attachment of the joining path to* K_4 *with an ear of length 2 is one of the vertices of degree 3 in* K_4 *with an ear of length 2.*

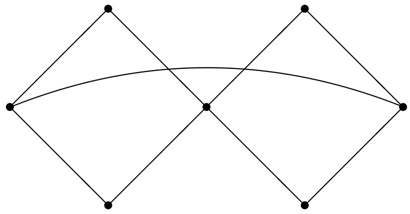
We have found some more Hall-2^+-critical graphs.

Theorem 6 *The following graphs are Hall-*2^+*-critical:*

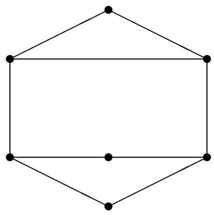
1. C_3 with two ears on adjacent edges,

2. C_4 with a triangle based on one edge and an ear of length m, $m \geq 2$, attached to non-adjacent vertices.

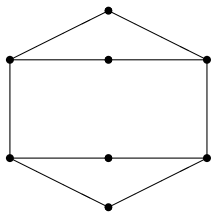
3. $G_6 =$



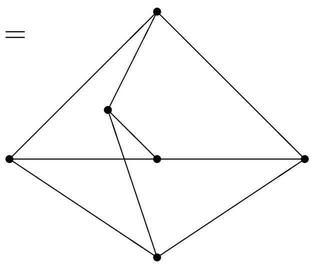
4. $G_7 =$



5. $G_8 =$



6. $G_9 =$



Proof: To prove that a graph G is Hall-2^+-critical it is necessary to prove that $h(G - v) \leq 2$ for every $v \in V(G)$, and that $h(G) > 2$. The first part is left to the reader: he should refer to Theorems 1, 3 and 4. To show that $h(G) > 2$ we must find a list assignment L such that

- $|L(v)| \geq 2$ for all $v \in V(G)$,

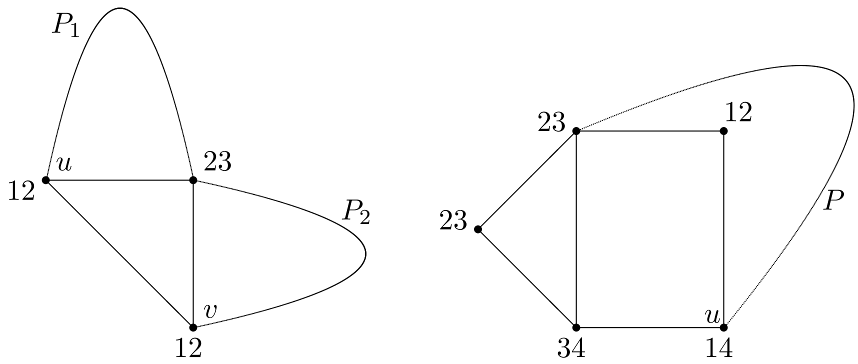


Figure 2: C_3 with two ears and C_4 with a triangle and an ear

- L and G satisfy Hall's condition, and
- G does not have a proper L-colouring.

We shall give list assignments (in word form), but then it is again left to the reader to check that the graphs and lists satisfy Hall's condition but do not permit a colouring. Note that a Hall-2^+-critical graph G, satisfies Hall's condition if and only if (1) holds with $H = G$, and the graphs $G - v$, for all $v \in V(G)$, have a proper L-colouring.

For C_3 with two ears see Figure 2. We must describe the list assignments for the internal vertices of the ears. If P_1 has odd length, then the vertex adjacent to u has assignment 12, the next vertex along the path has assignment 23, and any remaining vertices have assignment 13. If P_2 has odd length, then the vertex adjacent to v has assignment 12, the next vertex along the path has assignment 23, and any remaining vertices have assignment 13. If P_1 or P_2 has even length, then every vertex has assignment 13.

For C_4 with a triangle based on one edge and an ear attached to non-adjacent vertices see Figure 2 again. We describe the list assignments for the internal vertices of the ear. If P is of odd length, then the vertex adjacent to u has assignment 12, the next vertex has assignment 23, and any remaining vertices have assignment 13. If P is of even length, then every internal vertex on that ear has assignment 13.

For G_6, G_7, G_8 and G_9 see Figure 3. $\square$

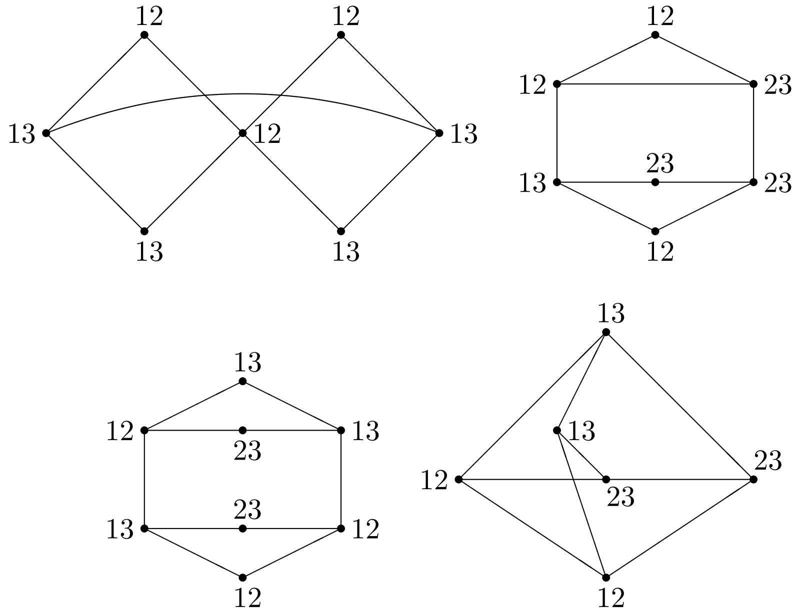


Figure 3: G_6, G_7, G_8 and G_9

Theorem 7 [7] *Except for K_4, every partial wheel graph with 3 or more radial edges has Hall number greater than 2.*

Theorem 7 follows easily from Theorem 5, but it is useful to state it separately since when trying to prove that a graph has Hall number greater than 2, it can sometimes be difficult to pick out an induced Hall-2^+-critical subgraph, while finding an induced partial wheel graph may be straightforward.

We shall prove Theorem 2 by showing that every graph's core has Hall number 1 or 2, or has an induced subgraph that is one of the Hall-2^+-critical graphs listed in Theorems 5 and 6. Thus we claim to have a complete list of Hall-2^+-critical graphs.

3 Graphs with one block in their core

We shall show that the only 2-connected graphs with Hall number 2 are

1. C_n, $n \geq 4$,
2. $\theta(m, 2, 1)$, $m \geq 2$,
3. $\theta(m, 2, 2)$, $m \geq 2$,
4. $\theta(3, 3, 2)$,
5. K_4 with an ear of length 2,
6. any cycle of even length with two triangles based on non-adjacent edges of the cycle.

Combined with Theorems 3 and 4, this will give a complete characterization of graphs with one block in their core with Hall number 2.

We need a lemma.

Lemma 8 *Let H be an induced subgraph of a 2-connected graph G where $|V(H)| \geq 2$. If $G \neq H$, then there exists an induced subgraph of G, H', such that either*

1. *H' is H with an ear of length p, $p \geq 3$, or*
2. *H' is H plus another vertex adjacent to q vertices in H, $q \geq 2$.*

Equivalently, we can write $p \geq 2$ and $q \geq 3$.

Proof: If $G \neq H$, then there is a vertex $v \in V(G) \setminus V(H)$ that is adjacent to at least one vertex in H. If v is adjacent to more than one vertex in H, then we let H' be the subgraph of G induced by $V(H) \cup \{v\}$. If v is adjacent to exactly one vertex x of H, then there must be at least one path from v to another vertex of H else $G - x$ is not connected. Let P be the shortest such path. Let u be the last vertex not in H as we move along P from v to H. If u is adjacent to more than one vertex in H, then we let H'

be the subgraph of G induced by $V(H) \cup \{u\}$. If u is adjacent to exactly one vertex in H, then we let H' be the subgraph of G induced by $V(H) \cup V(P)$. This is H with an ear since if a vertex on the path other than u and v was adjacent to a vertex of H, then P would not be the shortest possible path. □

Let G be a 2-connected graph. We know that G has Hall number 1 only if it is a clique. We must prove that if G it is not a clique or one of the graphs listed at the start of the section, then it has Hall number greater than 2. To do this we show that we can find an induced subgraph of G that is one of the graphs with Hall number greater than 2 listed in Theorems 5, 6, and 7. We call these graphs *forbidden*.

As G is 2-connected it has an induced subgraph that is a cycle. If G is a cycle, then $h(G) \in \{1, 2\}$. Suppose that G is not a cycle. Then by Lemma 8 we can find another induced subgraph of G that is either a cycle with an ear of length at least 2, or a cycle with a further vertex joined to 3 or more vertices of the cycle. In the former case the induced subgraph is $\theta(m_1, m_2, m_3)$ which is forbidden unless it is $\theta(3, 3, 2)$, $\theta(m, 2, 2)$, $m \geq 2$, or $\theta(m, 2, 1)$, $m \geq 2$ (Theorem 5.1). In the latter case the induced subgraph is $W(a_1, \ldots, a_k)$, $k \geq 3$, which is forbidden unless it is K_4 (Theorem 7). Therefore G has Hall number greater than 2 except possibly if it has an induced subgraph H where

1. $H = K_4$,
2. $H = \theta(3, 3, 2)$
3. $H = \theta(m, 2, 2)$, $m \geq 2$, or
4. $H = \theta(m, 2, 1)$, $m \geq 2$.

We consider the four cases separately.

Case 1: $H = K_4$.

If $G = K_4$, then $h(G) = 1$. Suppose that $G \neq K_4$. Then by Lemma 8 we can find an induced subgraph of G, H', that is either K_4 with an ear of length p, $p \geq 3$, or K_4 with one further vertex joined to 2 or more of the vertices of K_4. In the first case H' is forbidden (Theorem 5.4). In the latter case H' is K_4 with an ear of length 2, K_5 minus an edge or K_5, when the further vertex is joined to 2, 3 or 4 of the vertices of K_4 respectively. As K_5 minus an edge is forbidden (Theorem 5.7), G has Hall number greater than 2 except possibly if it has an induced subgraph H' where

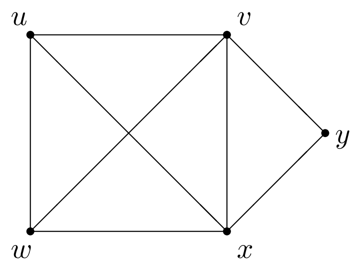
1. $H' = K_4$ with an ear of length 2, or

2. $H' = K_5$.

We consider the two subcases separately.

Subcase 1a: $H' = K_4$ with an ear of length 2.

If $G = H'$, then $h(G) = 2$. We shall show that if $G \neq H'$, then G has an induced subgraph with Hall number greater than 2. By Lemma 8 we can find an induced subgraph of G, H'', that is either H' with an ear of length p, $p \geq 3$, or H' with a further vertex joined to 2 or more vertices of H'. Let the vertices of H' be labelled as shown below.



Suppose that H'' is H' with an ear of length at least 3. If the ear is attached to 2 vertices other than y, then the subgraph of H'' induced by every vertex but y is K_4 with an ear of length greater than 2 which is forbidden (Theorem 5.4). Up to isomorphism, there are two other ways of attaching the ear. If it is attached to u and y, then the subgraph of H'' induced by every vertex but w is $W(p, 1, 1)$ which is forbidden (Theorem 5.13). If the ear is attached to v and y, then the subgraph of H'' induced by every vertex but x is *cuff* $(C_{p+1}, C_3, 0)$ which is forbidden (Theorem 5.23).

Suppose that H'' is H' with a further vertex z joined to between 2 and 5 of the vertices of H'. Suppose that z is joined to 2 vertices of H'. If neither of these is y, then H'' is K_4 with 2 disjoint ears each of length 2 which is forbidden (Theorem 5.5). If z is joined to 2 vertices including y, then there are 2 possibilities for the other vertex: u and v. If z is joined to u and y, then the subgraph of H'' induced by every vertex but w is $W(2,1,1)$ which is forbidden (Theorem 5.13). If z is joined to v and y, then the subgraph of H'' induced by every vertex but u is C_3 with 2 ears on adjacent edges which is forbidden (Theorem 6.1).

Suppose that z is joined to 3 vertices of H'. If none of these vertices is y, then the subgraph of H'' induced by every vertex but y is K_5 minus an edge which is forbidden (Theorem 5.7). If z is joined to y and 2 other vertices, then there are three possible choices for this pair: u and w; v and x; and u and v. If z is joined to u, w and y, then the subgraph of H'' induced by every vertex but w is $W(2,1,1)$ which is forbidden (Theorem 5.13). If z is joined to v, x and y, then H'' is 2 K_4's intersecting in a K_2 which is forbidden (Theorem 5.6). If z is joined to u, v and y, then the subgraph of H'' induced by every vertex but w is $W(1,1,1,1)$ which is forbidden (Theorem 5.14).

Suppose that z is joined to 4 vertices of H'. If z is joined to every vertex except y, then H'' is K_5 with an ear of length 2 which is forbidden (Theorem 5.3). If z is joined to every vertex except u, then the subgraph of H'' induced by every vertex but y is K_5 minus an edge which is forbidden (Theorem 5.7). If z is joined to every vertex except v, then the subgraph of H'' induced by every vertex but x is $W(2,1,1)$ which is forbidden (Theorem 5.13).

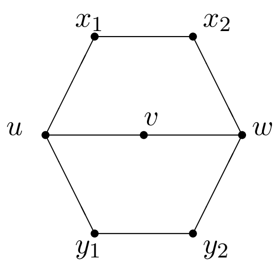
Finally suppose that z is joined to all 5 vertices of H'. Then the subgraph of H'' induced by every vertex but u is K_5 minus an edge which is forbidden (Theorem 5.7).

Subcase 1b: $H' = K_5$.

We prove by that the only graphs G which have Hall number not greater than 2 and have K_m, $m \geq 5$, as an induced subgraph are cliques. Let K_m, $m \geq 5$, be a largest clique in G. If $G \neq K_m$, then by Lemma 8 we can find another induced subgraph of G, H'', that is either K_m with an ear of length at least 2 or K_m with a further vertex joined to q vertices of K_m, $q \geq 3$. In the first case either K_4 with an ear of length at least 3 or K_5 with an ear of length 2 is an induced subgraph of H''; both are forbidden (Theorems 5.3 and 5.4). In the second case if $q < m$, then K_5 minus an edge is an induced subgraph of H'' which is forbidden (Theorem 5.7). If $q = m$, then $H'' = K_{m+1}$, contrary to the assumption that K_m is a largest clique in G.

Case 2: $H = \theta(3, 3, 2)$.

If $G = \theta(3, 3, 2)$, then $h(G) = 2$. We shall show that if $\theta(3, 3, 2)$ is an induced subgraph of G but $G \neq \theta(3, 3, 2)$, then $h(G) > 2$. By Lemma 8 we can find an induced subgraph H' that is either $\theta(3, 3, 2)$ with an ear of length p, $p \geq 3$, or $\theta(3, 3, 2)$ plus a further vertex joined to 2 or more vertices of $\theta(3, 3, 2)$. We label the vertices of $\theta(3, 3, 2)$ as shown below.



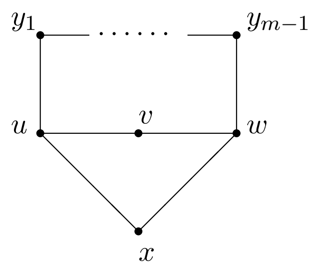
Suppose that H' is $\theta(3, 3, 2)$ with an ear of length p, $p \geq 3$. If this ear is not joined to v, then the subgraph of H' induced by every vertex but v is $\theta(p, 6 - r, r)$, $1 \leq r \leq 3$, which is forbidden (Theorem 5.1). If the ear is joined to v, then there are 2 possibilities for the other vertex of attachment: u and x_1. If the ear is attached to u and v, then the subgraph of H' induced by every vertex but y_1 and y_2 is $\theta(p, 4, 1)$ which is forbidden (Theorem 5.1). If the ear is attached to v and x_1, then the subgraph of H' induced by every vertex but x_2 is $\theta(p + 1, 4, 1)$ which is forbidden (Theorem 5.1).

Suppose that there is a vertex z joined to between 2 and 7 of the vertices of $\theta(3,3,2)$. If z is joined to 2 vertices, then there are 8 ways to choose this pair: u and v; u and w; u and x_1; u and x_2; v and x_1; x_1 and x_2; x_1 and y_1; and x_1 and y_2. In four of these cases H' is G_2, G_3, G_4 or G_5 which are all forbidden (Theorems 5.19-5.22). We consider the other cases. If z is joined to u and w, then H' is $\theta(3,3,2,2)$ which is forbidden (Theorem 5.2). If z is joined to u and x_1, then the subgraph of H' induced by every vertex but x_2 is $\mathit{Cuff}(C_5, C_3, 0)$ which is forbidden (Theorem 5.23). If z is joined to v and x_1, then the subgraph of H' induced by every vertex but x_2 is $\theta(4,3,1)$ which is forbidden (Theorem 5.1). If z is joined to x_1 and y_1, then the subgraph of H' induced by every vertex but x_2 is $\theta(4,3,1)$ which is forbidden (Theorem 5.1).

If z is joined to 3 or more vertices other than v, then the subgraph of H' induced by every vertex but v is a partial wheel graph with 3 or more radial edges and is not K_4 so it is forbidden (Theorem 7). The only remaining case is when z is joined to 3 vertices including v. The other two vertices must be x_i and y_j: if z is not joined to either y_1 or y_2, then the subgraph of H' induced by every vertex but y_1 and y_2 is $W(a_1, a_2, a_3)$ (which is not K_4 since $a_1 + a_2 + a_3 = 5$) which is forbidden (Theorem 7); a similar argument holds if z is not joined to either x_1 or x_2. There are two possibilities: $i = j$ and $i \neq j$. If z is joined to v, x_1 and y_1, then the subgraph of H' induced by every vertex but x_2 is $WL(2,1,1;1,3,1)$ which is forbidden (Theorem 5.17). If z is joined to v, x_1 and y_2, then the subgraph of H' induced by every vertex but x_2 and y_1 is $\theta(3,3,1)$ which is forbidden (Theorem 5.1).

Case 3: $H = \theta(m,2,2)$, $m \geq 2$.

If $G = \theta(m,2,2)$, then $h(G) = 2$. We shall show that if $\theta(m,2,2)$ is an induced subgraph of G but $G \neq \theta(m,2,2)$, then $h(G) > 2$. By Lemma 8, we can find an induced subgraph H' that is either $\theta(m,2,2)$ with an ear of length p, $p \geq 2$, or $\theta(m,2,2)$ with a further vertex joined to 3 or more vertices of $\theta(m,2,2)$. Let the vertices of $\theta(m,2,2)$ be labelled as shown below.



Suppose that H' is $\theta(m, 2, 2)$ with an ear of length p, $p \geq 2$. There are six ways to choose a pair of vertices to which to attach the ear: u and v; u and w; v and x; u and y_i; v and y_i; and y_i and y_j, $i < j$.

If the ear is attached to u and v, then the subgraph of H' induced by every vertex but x is $\theta(m+1, p, 1)$ which is forbidden unless $p = 2$ (Theorem 5.1). If $p = 2$, then H' is C_4 with a triangle based on one edge and an ear attached to non-adjacent vertices which is forbidden (Theorem 6.2).

If the ear is attached to u and w, then the subgraph of H' induced by every vertex but x is $\theta(m, p, 2)$ which is forbidden unless $\{m, p\} \in \{\{2\}, \{2, 3\}, \{3\}\}$ (Theorem 5.1). If $m = 2$ (or $p = 2$), then H' is $\theta(p, 2, 2, 2)$ (or $\theta(m, 2, 2, 2)$), which is forbidden (Theorem 5.2). If $m = p = 3$, then H' is $\theta(3, 3, 2, 2)$ which is forbidden (Theorem 5.2).

If the ear is attached to v and x, then H' is $WL(m, 1, 1; 1, p, 1)$ which is forbidden (Theorem 5.17).

If the ear is attached to u and y_i, then unless $i = m-1$ the subgraph of H' induced by every vertex but y_j, for all $j > i$, is *cuff* $(C_{p+i}, C_4, 0)$ which is forbidden (Theorem 5.23). If $i = m-1$, then the subgraph of H' induced by every vertex but x is $\theta(p, m-1, 3)$ which is forbidden unless $\{p, m-1\} \in \{\{1, 2\}, \{2\}, \{2, 3\}\}$ (Theorem 5.1). If $\{p, m-1\} = \{1, 2\}$, then H' is C_4 with a triangle based on one edge and an ear attached to non-adjacent vertices, which is forbidden (Theorem 6.2). If $p = m-1 = 2$, then H' is G_6 which is forbidden (Theorem 6.3). If $\{p, m-1\} = \{2, 3\}$, then H' is G_2 which is forbidden (Theorem 5.19).

If the ear is attached to v and y_i, then unless $i = m-1$ the subgraph of H' induced by every vertex but y_j, for all $j > i$, is $\theta(p+i, 3, 1)$ which is forbidden (Theorem 5.1). If $i = m-1$, then unless $i = 1$ the subgraph of

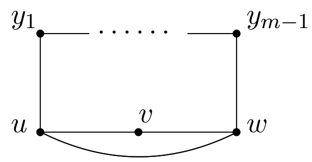
H' induced by every vertex but y_j, for all $j < i$, is $\theta(p+1, 3, 1)$ which is forbidden (Theorem 5.1). If $i = m-1 = 1$, then H' is $WL(p, 1, 1; 1, 2, 1)$ which is forbidden (Theorem 5.17).

If the ear is attached to y_i and y_j, then the subgraph of H' induced by every vertex but x is $\theta(p, j-i, m+2+i-j)$ which is forbidden unless $p = 2$ and $j - i \in \{1, 2\}$ (since $m + 2 + i - j \geq 4$) (Theorem 5.1). Unless $i = 1$, the subgraph of H' formed by every vertex but y_k, for all $k < i$, is *cuff* $(C_4, C_{p+j-i}, m-j)$, and unless $j = m-1$, the subgraph formed by every vertex but y_k, for all $k > j$, is *cuff* (C_4, C_{p+j-i}, i); both are forbidden (Theorem 5.23). So the only remaining cases are $p = 2$, $i = 1$, $j = m-1$ and $j - i \in \{1, 2\}$. In these cases H' is either G_7 or G_8 which are both forbidden (Theorems 6.4 and 6.5)

Suppose that H' is $\theta(m, 2, 2)$ with another vertex z joined to 3 or more vertices of $\theta(m, 2, 2)$. If z is joined to three or more vertices of $\theta(m, 2, 2)$ other than x (or equivalently v), then the subgraph of H' induced by every vertex but x (or v) is a partial wheel graph with 3 or more radial edges and is not K_4 so is forbidden (Theorem 7). The only remaining case is when z is joined to v, x and one other vertex. There are 2 possibilities for the other vertex: u and y_i. If z is joined to u, v and x, then the subgraph of H' induced by every vertex but y_i, for all i, is $W(2, 1, 1)$ which is forbidden (Theorem 5.13). If z is joined to v, x and y_i, then the subgraph induced by every vertex but x is $\theta(2, i+1, m+1-i)$ which is forbidden unless $\{i+1, m+1-i\} \in \{\{2\}, \{2, 3\}, \{3\}\}$. If $i+1 = m+1-i = 3$, then, as in Case 2, $H' - y_1 = WL(2, 1, 1; 1, 3, 1)$, which is forbidden (Theorem 5.17). If $\{i+1, m+1-i\} = \{2, 3\}$, then we can assume that $i = 1$ (and $m = 3$) and so the subgraph of H' induced by every vertex but y_2 is $WL(2, 1, 1; 1, 2, 1)$ which is forbidden (Theorem 5.17). If $i+1 = m+1-i = 2$, then H' is G_9 which is forbidden (Theorem 6.6).

Case 4: $H = \theta(m, 2, 1)$, $m \geq 2$.

If $G = \theta(m, 2, 1)$, then $h(G) = 2$. We shall show that if $\theta(m, 2, 1)$ is an induced subgraph of G but $G \neq \theta(m, 2, 1)$, then $h(G) > 2$ or G is either a cycle of even length with two triangles based on non-adjacent edges or K_4 with an ear of length 2. By Lemma 8, we can find an induced subgraph of G, H', that is either $\theta(m,2,1)$ with an ear of length p, $p \geq 2$, or $\theta(m,2,1)$ with a further vertex joined to 3 or more vertices of $\theta(m,2,1)$. Let the vertices of $\theta(m,2,1)$ be labelled as shown below.



Suppose that H' is $\theta(m,2,1)$ with an ear of length p, $p \geq 2$. There are five ways to choose a pair of vertices to which to attach the ear: u and v; u and w; u and y_i; v and y_i; y_i and y_j, $i < j$.

If the ear is attached to u and v, then H' is C_3 with 2 ears on adjacent edges which is forbidden (Theorem 6.1).

If the ear is attached to u and w, then the subgraph of H' induced by every vertex but v is $\theta(p,m,1)$ which is forbidden if $p \geq 3$ and $m \geq 3$. If $p = 2$ (or $m = 2$), then H' is $\theta(m,2,2,1)$ (or $\theta(p,2,2,1)$), which is forbidden (Theorem 5.2).

If the ear is attached to u and y_i, then unless $i = m-1$ the subgraph of H' induced by every vertex but y_j, for all $j > i$, is *cuff*$(C_{p+i}, C_3, 0)$ which is forbidden unless $p + i = 3$ (Theorem 5.23). If $p + i = 3$, then H' is a cycle with 2 triangles based on adjacent edges which is forbidden (Theorem 5.8). If $i = m-1$, then the subgraph of H' induced by every vertex but v is $\theta(m-1, p, 2)$ which is forbidden unless $\{m-1, p\} \in \{\{1,2\},\{1,3\},\{2\},\{2,3\},\{3\}\}$ (Theorem 5.1). If $m - 1 = p = 3$, then H' is G_5, which is forbidden (Theorem 5.22). If $\{m-1, p\} \in \{\{2\},\{2,3\}\}$, then H' is a forbidden graph of the type described in Theorem 6.2. If $m - 1 = 1$, then H' is C_3 with 2 ears on adjacent edges which is forbidden (Theorem 6.1).

If the ear is attached to v and y_i, then H' is $WL(p, i, m-i; 1, 1, 1)$ which is forbidden (Theorem 5.15).

If the ear is attached to y_i and y_j and $p = 2$ and $j - i = 1$, then H' is a cycle with 2 triangles on non-adjacent edges. If the cycle is of odd length, then H' is forbidden (Theorem 5.10); if the cycle is of even length, then $h(H') = 2$ but we shall show below in Subcase 4a that either $G = H'$ or $h(G) > 2$. If $p \neq 2$ or $j - i \neq 1$, then $p + j - i \geq 4$. Unless $j = m - 1$ the subgraph of H' induced by every vertex but y_k, for all $k > j$, is *cuff* (C_{p+j-i}, C_3, i), and unless $i = 1$ the subgraph of H' induced by every vertex but y_k, for all $k < i$, is *cuff* $(C_{p+j-i}, C_3, m - j)$; both are forbidden (Theorem 5.23). If $i = m - j = 1$, then the subgraph induced by every vertex but v is $\theta(3, p, j - i)$ which is forbidden unless $\{p, j - i\} \in \{\{2\}, \{2, 3\}\}$ (Theorem 5.1). If $p = j - i = 2$, then H' is G_7 which is forbidden (Theorem 6.4). If $\{p, j - i\} = \{2, 3\}$, then H' is $\theta(3, 3, 2)$ with an ear which is forbidden (Case 2).

Suppose that H' is $\theta(m, 2, 1)$ with another vertex z joined to 3 or more vertices of $\theta(m, 2, 1)$. If z is joined to 3 vertices other than v, then the subgraph of H' induced by every vertex but v is a partial wheel graph with 3 or more radial edges and is forbidden unless it is K_4 (Theorem 7). If it is K_4, then H' is K_5 minus an edge, which is forbidden (Theorem 5.7), or H' is K_4 with an ear of length 2 and either $G = H'$ or $h(G) > 2$ (Subcase 1a). The only remaining case is when z is joined to v and 2 other vertices. There are 3 possible choices for this pair: u and w; u and y_i; and y_i and y_j, $i < j$.

If z is joined to u, v and w, then H' is K_4 with an ear of length m. If $m > 2$, then H' is forbidden (Theorem 5.4), else we refer again to Subcase 1a.

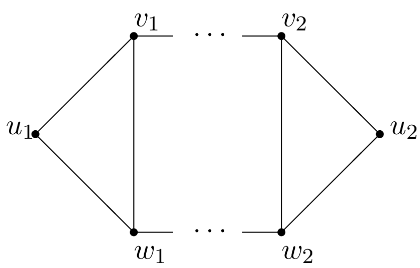
If z is joined to u, v and y_i, then unless $i = m - 1$ the subgraph of H' induced by every vertex but y_k, for all $k > i$, is C_3 with 2 ears on adjacent edges which is forbidden (Theorem 6.1), and unless $i = 1$ the subgraph of H' induced by every vertex but y_k, for all $k < i$, is also C_3 with 2 ears on adjacent edges. The remaining case is when $i = m - 1 = 1$. In this case $H' = W(1, 1, 1, 1)$ which is forbidden (Theorem 5.14).

If z is joined to v, y_i and y_j, then unless $i = j - 1$ the subgraph of H' induced by every vertex but y_k, for all k, $i < k < j$, is $W(1 + i, m + 1 - j, 1)$ which is forbidden (Theorem 5.13). Suppose that $i = j - 1$. If either $i = 1$

or $m-j=1$, or i and $m-j$ are both odd, then $H' = G_1$ which is forbidden (Theorem 5.12). If $i \neq 1$ and $m-j \neq 1$ and i is even (or $m-j$ is even), then the subgraph of H' induced by every vertex but y_k, for all $k > j$, is a cycle of odd length with 2 triangles on non-adjacent edges which is forbidden (Theorem 5.10).

Subcase 4a: H' is a cycle of even length with 2 triangles on non-adjacent edges.

If $G = H'$, then $h(G) = 2$. We shall show that if H' is an induced subgraph of G but $G \neq H'$, then $h(G) > 2$. By Lemma 8, we can find an induced subgraph H'' that is either H' with an ear of length p, $p \geq 2$, or H' with a further vertex joined to 3 or more vertices of H'. Let the vertices be labelled as shown below.



Let the lengths of the paths between v_1 and v_2 and between w_1 and w_2 be p_1 and p_2. Note that p_1 and p_2 have the same parity.

Suppose that H'' is H' with an ear. If the ear is not attached to both u_1 and u_2, then the subgraph, J, of H'' induced by every vertex but u_1 (or u_2) will be a graph considered in Case 4, and so it will be forbidden unless $J = H'$ in which case H'' is either a cycle of even length with 3 triangles on non adjacent edges or it has an induced subgraph that is a cycle with 2 triangles on adjacent edges or on the same edge; all these graphs are forbidden (Theorems 5.8, 5.9 and 5.11). If the ear is attached to both u_1 and u_2, then if p has the same parity as p_1 and p_2, $H'' = G_1$ which is forbidden (Theorem 5.12); otherwise the subgraph induced by every vertex but the internal vertices of the path between v_1 and v_2 is an odd cycle with

2 triangles on non-adjacent edges which is forbidden (Theorem 5.10).

If H'' is H' with a further vertex joined to 3 or more vertices of H', then let J and J' be the subgraphs of H'' induced by every vertex but u_1 and u_2 respectively. It is easy to see that either J or J' is a graph that was shown to be forbidden in Case 4.

4 Graphs with more than one block in their core

In this section we complete the proof of Theorem 2 by proving that a graph with at least two blocks in its core has Hall number 2 if and only if every block is either a clique, $\theta(2,2,1)$ or K_4 with an ear of length 2; at least one block is not a clique; and if a block is $\theta(2,2,1)$ or K_4 with an ear, then only the vertices of degree 2 can be cutvertices in the core of the graph.

First we prove that there is no other graph with at least 2 blocks in its core G with Hall number 2. Each induced subgraph of G, and therefore each block of G, must be a graph with Hall number at most 2. Suppose that a block of G contains an induced cycle C of length m_1, $m_1 \geq 4$. Let P be the shortest path that joins C to an induced cycle in another block. Let the other cycle be D and have length m_2, let $P = v_1 v_2 \cdots v_{r-1} v_r$ where $v_1 \in V(C)$, $v_r \in V(D)$. Let H be the subgraph induced by $V(C) \cup V(P) \cup V(D)$. For $3 \leq i \leq r-2$, v_i is not adjacent to a vertex in either $V(C)$ or $V(D)$ else we can find a shorter path than P. Also v_{r-1} is adjacent to only one vertex, v_r, in $V(D)$ since if it is also adjacent to a vertex $w \in V(D)$, then instead of D we can consider the cycle induced by v_{r-1} and the vertices on D between v_r and w inclusive and again obtain a shorter path than P. If v_2 is adjacent only to v_1 in $V(C)$, then H is *Cuff* (C_{m_1}, C_{m_2}, l), $m_1 \geq 4$, $m_2 \geq 3$, $l \geq 0$, which is forbidden (Theorem 5.23). Suppose that v_2 is adjacent to more than one vertex in $V(C)$. Then we can find a cycle of length m_1' that is induced by v_2 and some of the vertices on C. So if $m_2 > 3$, then we can find an induced *Cuff* $(C_{m_2}, C_{m_1'}, l)$, $m_2 \geq 4$, $m_1' \geq 3$, $l \geq 0$. Hence we shall assume that $D = K_3$. If v_2 is adjacent to 2 vertices, u and v_1, in $V(C)$ and they are adjacent, then H is *Cuff* $(\theta(m_1-1,2,1), K_3, l)$, $m_1 - 1 \geq 3$, $l \geq 0$, where the point of attachment of the joining path to $\theta(m_1-1,2,1)$

is the vertex of degree 2 that is adjacent to the 2 vertices of degree 3, which is forbidden (Theorems 5.25); if u and v_1 are not adjacent, then the graph induced by the vertices on C between u and v_1 inclusive and $V(P)$ and $V(D)$ is *Cuff* $(C_{m'_1}, C_{m_2}, l)$, $m'_1 \geq 4$, $m_2 \geq 3$, $l \geq 0$. If v_2 is adjacent to $\{v_1, u_1, \ldots, u_s\} \subseteq V(D)$, $s \geq 2$, then we can assume that $v_1 u_1 \ldots u_s$ is a path (else again we can find an induced *Cuff* $(C_{m'_1}, C_{m_2}, l)$, $m'_1 \geq 4$, $m_2 \geq 3$, $l \geq 0$). Therefore the graph induced by v_1, u_1, u_2, $V(P)$ and $V(D)$ is *Cuff* $(\theta(2,2,1), K_3, l)$, $l \geq 0$, where the point of attachment of the path of the joining path to $\theta(2,2,1)$ is one of the vertices of degree 3, which is forbidden (Theorem 5.24). We have shown that if a block of G contains an induced cycle of length greater than 3, then G has Hall number greater than 2. By Theorem 1 and Section 3, each block must be a clique, $\theta(2,2,1)$ or K_4 with an ear of length 2; these are the only 2-connected graphs with Hall number at most 2 and no induced cycle of length greater than 3. If $\theta(2,2,1)$ or K_4 with an ear of length 2 is a block, then only the vertices of degree 2 can be cutvertices in G else *Cuff* $(\theta(2,2,1), K_3, l)$, $l \geq 0$, where the point of attachment of the joining path to $\theta(2,2,1)$ is one of the vertices of degree 3, or *Cuff* (K_4 with an ear of length 2, K_3, l), $l \geq 0$, where the point of attachment of the joining path to K_4 with an ear is one of the vertices of degree 3, is an induced subgraph; both are forbidden (Theorems 5.24 and 5.26). Finally, we insist that at least one block is not a clique else $h(G) = 1$ (Theorem 1).

We shall call every vertex that cannot be a cutvertex in the core a *non-attachment vertex*; the other vertices are *attachment vertices*.

To prove that the cores listed at the start of the section have Hall number 2 we shall prove a slightly stronger result.

Theorem 9 *Let G be the core of a graph. Let G have at least two blocks where every block is either a clique, $\theta(2,2,1)$ or K_4 with an ear of length 2, and if a block is $\theta(2,2,1)$ or K_4 with an ear, then the vertices not of degree 2 are not cutvertices. Let L be a list assignment for G such that $|L(v)| \geq 1$ for each attachment vertex v, $|L(v)| \geq 2$ for each non-attachment vertex v, and G and L satisfy Hall's condition. Then G has a proper L-colouring.*

Proof: First we shall show that we can give any block B of G a proper L-colouring. If B is a clique, then this follows from Theorem 1. Suppose that B is $\theta(2,2,1)$ or K_4 with an ear of length 2. If $|L(v)| \geq 2$ for each vertex v, then there is a proper L-colouring since both graphs have Hall number 2. Suppose that for some vertex v, which must be a vertex of degree 2, $|L(v)| = 1$, that is, say, $L(v) = \{1\}$. We shall show that if G has no proper L-colouring, then B and L do not satisfy Hall's condition. Let x and y be the vertices of B adjacent to v. As $B \setminus \{v\}$ is a clique it has a proper L-colouring. Furthermore, in every proper L-colouring of $B \setminus \{v\}$, 1 must be used to colour either x or y else we can obtain a colouring for B by using 1 on v. Define a list assignment L' for $B \setminus \{v\}$ as follows

$$L'(u) = \begin{cases} L(u) & \text{if } u \notin \{x,y\} \\ L(u) \setminus \{1\} & \text{if } u \in \{x,y\}. \end{cases}$$

Thus $B \setminus \{v\}$ has no proper L'-colouring and so $B \setminus \{v\}$ and L' cannot satisfy Hall's condition: there is a subgraph H' of $B \setminus \{v\}$ such that

$$|V(H')| > \sum_{\sigma} \alpha(\sigma, L', H')$$

and for all $u \in V(H') \setminus \{x,y\}$, $1 \notin L'(u)$ else $|V(H')| > \sum_\sigma \alpha(\sigma, L, H')$; a contradiction. Therefore $H = H' \cup \{v\}$ and L do not satisfy Hall's condition since $|V(H)| = |V(H')| + 1$ and $\sum_\sigma \alpha(\sigma, L, H) = \sum_\sigma \alpha(\sigma, L', H') + 1$.

We shall show that if G_1 and G_2 are graphs that each satisfy the conditions of the theorem, then so is G', the graph formed if G_1 and G_2 intersect in a single vertex x that is an attachment vertex in each of G_1 and G_2. Thus by induction the theorem will be proved.

Let L be a list assignment such that $|L(v)| \geq 1$ for each attachment vertex v of G' and $|L(v)| \geq 2$ for each non-attachment vertex v of G'. We shall show that if G' cannot be coloured, then G' and L do not satisfy Hall's condition. Let $L(x) = \{1, \ldots, t\}$. If either G_1 or G_2 has no proper L-colouring, then G and L do not satisfy Hall's condition. Otherwise let $\{1, \ldots, k\}$ be the colours that are not used on x in any proper L-colouring of G_1; so $k < t$ as G_1 can be coloured. If at least one colour in the set $\{k+1, \ldots, t\}$ is used on x in a proper L-colouring of G_2, then we can colour

G'. Assume that there is no such colouring of G_2. We define two new list assignments, L_1 and L_2, for G_1 and G_2 respectively, as follows

$$L_1(u) = \begin{cases} L(u) & \text{if } u \neq x \\ \{1, \ldots, k\} & \text{if } u = x, \end{cases}$$

$$L_2(u) = \begin{cases} L(u) & \text{if } u \neq x \\ \{k+1, \ldots, t\} & \text{if } u = x. \end{cases}$$

As G_1 does not have a proper L_1-colouring, G_1 and L_1 do not satisfy Hall's condition so there must exist an induced subgraph, H_1, of G_1 such that

$$|V(H_1)| - 1 \geq \sum_{\sigma} \alpha(\sigma, L_1, H_1). \tag{2}$$

Note that $x \in V(H_1)$ else we would have $|V(H_1)| - 1 \geq \sum_{\sigma} \alpha(\sigma, L, H_1)$ and then G_1 could not be coloured with L.

Similarly, as G_2 does not have a proper L_2-colouring there must exist an induced subgraph, H_2, of G_2 such that $x \in V(H_2)$ and

$$|V(H_2)| - 1 \geq \sum_{\sigma} \alpha(\sigma, L_2, H_2). \tag{3}$$

Let $H = H_1 \cup H_2$. Clearly H is an induced subgraph of G. For each colour σ, it is easy to see that $\alpha(\sigma, L, H) \leq \alpha(\sigma, L_1, H_1) + \alpha(\sigma, L_2, H_2)$. Therefore

$$\sum_{\sigma} \alpha(\sigma, L, H) \leq \sum_{\sigma} \alpha(\sigma, L_1, H_1) + \sum_{\sigma} \alpha(\sigma, L_2, H_2),$$

and so by (2) and (3)

$$\begin{aligned} \sum_{\sigma} \alpha(\sigma, L, H) &\leq |V(H_1)| + |V(H_2)| - 2 \\ &= |V(H)| - 1. \end{aligned}$$

Therefore Hall's condition is not satisfied by G and L. $\square$

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