The computational complexity of the parallel knock-out problem

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Abstract. We consider computational complexity questions related to parallel knock-out schemes for graphs. In such schemes, in each round, each remaining vertex of a given graph eliminates exactly one of its neighbours. We show that the problem of whether, for a given graph, such a scheme can be found that eliminates every vertex is NP-complete. Moreover, we show that, for all fixed positive integers $k \geq 2$, the problem of whether a given graph admits a scheme in which all vertices are eliminated in at most k rounds is NP-complete. For graphs with bounded tree-width, however, both of these problems are shown to be solvable in polynomial time.

Keywords: parallel knock-out; graphs; computational complexity.

1 Introduction

In this paper, we consider *parallel knock-out schemes* for finite undirected simple graphs. These were introduced by Lampert and Slater [5]. Such a scheme proceeds in rounds: in the first round each vertex in the graph selects exactly one of its neighbours, and then all the selected vertices are eliminated simultaneously. In subsequent rounds this procedure is repeated in the subgraph induced by those vertices not yet eliminated. The scheme continues until there are no vertices left, or until an isolated vertex is obtained (since an isolated vertex will never be eliminated).

A graph is *reducible* if there exists a parallel knock-out scheme that eliminates the whole graph. The *parallel knock-out number* of a graph G, denoted by $\text{pko}(G)$, is the minimum number of rounds in a parallel knock-out scheme that eliminates every vertex of G. If G is not reducible, then $\text{pko}(G) = \infty$. Consider the following decision problem.

Parallel Knock-Out (PKO)
Instance: A graph G.
Question: Is G reducible?

In [5], it was claimed that PKO is NP-complete even when restricted to the class of bipartite graphs. No proof was given; the reader was referred to a paper that was in preparation. Our attempts to obtain and verify this proof have been unsuccessful. We shall obtain the result as a corollary to a stronger theorem (Theorem 1 below) by considering a related problem, which is defined for each positive integer k.

Parallel Knock-Out (k) (PKO(k))
Instance: A graph G.
Question: Is $\mathrm{pko}(G) \leq k$?

That there is a polynomial algorithm to decide PKO(1) follows easily from a piece of graph theory folklore (see [1] for details). Our first result classifies the complexity of PKO(k), $k \geq 2$.

Theorem 1. *For $k \geq 2$, PKO(k) is NP-complete even if instances are restricted to the class of bipartite graphs.*

In [1], it was shown, using a dynamic programming approach, that the parallel knock-out number for trees can be computed in polynomial time. It was asked whether this result could be extended to graphs with bounded tree-width. In our second result, we give an affirmative answer.

Theorem 2. *The problem PKO(k) can be solved in linear time on graphs with bounded tree-width.*

We will also show that PKO can be solved in polynomial time on graphs with bounded tree-width.

The paper is organised as follows. In the next two sections we introduce a number of definitions and simple results. In Section 4 and Section 5 are the proofs and corollaries of Theorems 1 and 2 respectively.

2 Preliminaries

Graphs in this paper are denoted by $G = (V, E)$. An edge joining vertices u and v is denoted uv. In the *null graph*, $V = E = \emptyset$. For graph terminology not defined below, refer to [2].

For a vertex $u \in V$ we denote its *neighbourhood*, that is, the set of adjacent vertices, by $N(u) = \{v \mid uv \in E\}$. The *degree* of a vertex is the number of edges incident with it, or, equivalently, the size of its neighbourhood.

For a graph G, a *KO-selection* is a function $f : V \to V$ with $f(v) \in N(v)$ for all $v \in V$. If $f(v) = u$, we say that vertex v *fires at* vertex u, or that vertex u *is knocked out* by vertex v.

For a KO-selection f, we define the corresponding *KO-successor* of G as the subgraph of G that is induced by the vertices in $V \setminus f(V)$; if H is the KO-successor of G we write $G \rightsquigarrow H$. Note that every graph without isolated vertices has at least one KO-successor. A graph G is called *KO-reducible*, if there exists a finite sequence

$$G \rightsquigarrow G_1 \rightsquigarrow G_2 \rightsquigarrow \cdots \rightsquigarrow G_r,$$

where G_r is the null graph. If no such sequence exists, then $\mathrm{pko}(G) = \infty$. Otherwise, the parallel knock-out number $\mathrm{pko}(G)$ of G is the smallest number r for which such a sequence exists. A sequence of KO-selections that transform G into the null graph is called a *KO-reduction scheme*. A single step in this sequence is called a *round* of the KO-reduction scheme. A subset of V is *knocked out* in a certain round if every vertex in the subset is knocked out in that round.

We make some simple observations that we will use later on.

Observation 1 *Let G be a graph on at least three vertices. If G contains two vertices of degree 1 that share the same neighbour, then G is not KO-reducible.*

Observation 2 *Let u_1, u_2, u_3, u_4 be four vertices of a KO-reducible graph G such that $N(u_2) = \{u_1, u_3\}$, $N(u_3) = \{u_2, u_4\}$ and $N(u_4) = \{u_3\}$. If u_1 is knocked out in the first round of a KO-reduction scheme, then u_1 fires at u_2 in the first round.*

An odd path $u_1u_2 \dots u_{2k+1}$ is called a *centred path* of G with *centrevertex* u_{k+1} if $G - \{u_{k+1}\}$ contains as components the path $u_1u_2 \dots u_k$ and the path $u_{k+2}u_{k+3} \dots u_{2k+1}$.

Observation 3 *Let $P = u_1u_2 \dots u_7$ be a centred path of a KO-reducible graph G. In the first round of any KO-reduction scheme u_1 and u_2 fire at each other, u_3 fires at u_2, u_6 and u_7 fire at each other, u_5 fires at u_6, u_4 fires at u_3 or u_5, and u_4 will not be knocked out. In the second round of any KO-reduction scheme u_4 and its remaining neighbour in P fire at each other.*

3 NP-complete problems

In this section, we consider two NP-complete problems that we will use in the proof of Theorem 1. We refer to [4] and [6] for further details.

DOMINATING SET (DS)
Instance: A graph $G = (V, E)$ and a positive integer p.
Question: Does G have a *dominating set* of size at most p, that is, is there a subset $V' \subseteq V$ such that $|V'| \leq p$ and every vertex of G is in V' or adjacent to a vertex in V'?

A *hypergraph* $J = (Q, \mathcal{S})$ is a pair of sets where $Q = \{q_1, \dots, q_m\}$ is the vertex set and $\mathcal{S} = \{S_1, \dots, S_n\}$ is the set of *hyperedges*. Each member S_j of $\mathcal{S}$ is a subset of Q.

HYPERGRAPH 2-COLOURABILITY (H2C)
Instance: A hypergraph $J = (Q, \mathcal{S})$.
Question: Is there a 2-*colouring* of $J = (Q, \mathcal{S})$, that is, a partition of Q into sets B and W such that, for each $S \in \mathcal{S}$, $B \cap S \neq \emptyset$ and $W \cap S \neq \emptyset$.

The *incidence graph* I of a hypergraph $J = (Q, \mathcal{S})$ is a bipartite graph with vertex set $Q \cup \mathcal{S}$ where (q, S) forms an edge if and only if $q \in S$.

With a hypergraph $J = (Q, \mathcal{S})$ we can associate another hypergraph $J' = (X, \mathcal{Z})$ called the *triple* of J; triples of hypergraphs will play a crucial role in our NP-completeness proofs in the next section. It requires a little effort to define the vertices X and hyperedges $\mathcal{Z}$ of the triple of J.

Recall that $Q = \{q_1, \ldots, q_m\}$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$. For $1 \le i \le m$, let $\ell(i)$ be the number of hyperedges in $\mathcal{S}$ that contain q_i, let $Q_i = \{q_i^1, \ldots, q_i^{\ell(i)}\}$ and let $U_i = \{u_i^1, \ldots, u_i^{\ell(i)}\}$. The union of all such sets is the vertex set of J', that is

$$X = \bigcup_{i=1}^{m} (Q_i \cup U_i).$$

Now the hyperedges:

- for $1 \le i \le m$, for $1 \le k \le \ell(i)$, let $P_i^k = \{q_i^k, u_i^k\}$,
- for $1 \le i \le m$, for $1 \le k \le \ell(i) - 1$, let $R_i^k = T_i^k = \{u_i^k, q_i^{k+1}\}$, and
- for $1 \le i \le m$, let $R_i^{\ell(i)} = T_i^{\ell(i)} = \{u_i^{\ell(i)}, q_i^1\}$.

Let $\mathcal{P}_i = \{P_i^1, \ldots, P_i^{\ell(i)}\}$, $\mathcal{R}_i = \{R_i^1, \ldots, R_i^{\ell(i)}\}$, and $\mathcal{T}_i = \{T_i^1, \ldots, T_i^{\ell(i)}\}$, and let

$$\mathcal{P} = \bigcup_{i=1}^{m} \mathcal{P}_i, \quad \mathcal{R} = \bigcup_{i=1}^{m} \mathcal{R}_i, \quad \mathcal{T} = \bigcup_{i=1}^{m} \mathcal{T}_i.$$

For $1 \le j \le n$, there is also a hyperedge S_j'. If in J, S_j contains q_i, then in J', S_j' contains a vertex of Q_i. In particular, if S_j is the kth hyperedge that contains q_i in J, then S_j' contains q_i^k. For example, if q_1 is in S_1, S_4 and S_7 in J, then $\ell(1) = 3$ and in J' there are vertices q_1^1, q_1^2, q_1^3 with $q_1^1 \in S_1'$, $q_1^2 \in S_4'$, and $q_1^3 \in S_7'$.

Let $\mathcal{S}' = \{S_1', \ldots, S_n'\}$. The set of hyperedges for J' is

$$\mathcal{Z} = \mathcal{S}' \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{T}.$$

We denote the incidence graph of the triple J' by I'. See Figure 1 for an example that illustrates the case where q_1 belongs to S_1, S_4 and S_7.

Proposition 1. *$J = (Q, \mathcal{S})$ has a 2-colouring $B \cup W$ if and only if $J' = (X, \mathcal{Z})$ has a 2-colouring $B' \cup W'$ such that for each $1 \le i \le m$ either $Q_i \subseteq B'$ and $U_i \subseteq W'$, or $Q_i \subseteq W'$ and $U_i \subseteq B'$.*

Proof. Suppose $B \cup W$ is a 2-colouring of J. Define a partition $B' \cup W'$ of X as follows. If q_i is in B, then each q_i^k is in B' and each u_i^k is in W'. If q_i is in W, then each q_i^k is in W' and each u_i^k is in B'. Obviously, $B' \cup W'$ is a 2-colouring of J' with the desired property.

Suppose we have a 2-colouring $B' \cup W'$ of J' such that for each $1 \le i \le m$ either $Q_i \subseteq B'$ and $U_i \subseteq W'$, or $Q_i \subseteq W'$ and $U_i \subseteq B'$. Then let $q_i \in B$ if and only if $Q_i \subseteq B'$, and let $W = Q \setminus B$. Clearly, if S_j contains only elements from B (respectively W), then S_j' would contain only elements from B' (respectively W'). Hence $B \cup W$ is a 2-colouring of J. $\square$

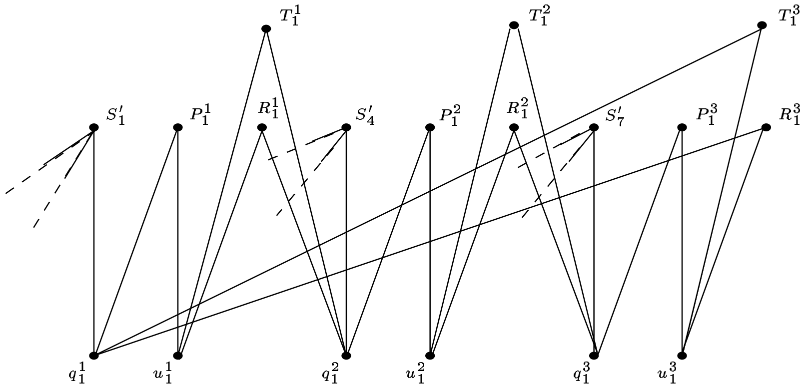

Fig. 1. Part of the incidence graph of the triple of a hypergraph.

4 Complexity classification

Theorem 1 *For $k \geq 2$, PKO(k) is NP-complete even if instances are restricted to the class of bipartite graphs.*

Proof. The proof is in three cases.

Case 1. $k = 2$. We use reduction from DS. Given $G = (V, E)$ and a positive integer $p \leq |V|$, we shall construct a bipartite graph B such that $\mathrm{pko}(B) = 2$ if and only if G has a dominating set D where $|D| \leq p$.

Let the vertex set of B be the disjoint union of $V = \{v_1, \ldots, v_n\}$, $V' = \{v_1', \ldots, v_n'\}$ and $W = \{w_1, \ldots, w_{n-p}\}$. Let the edge set of B contain

- $v_i v_i'$, $1 \leq i \leq n$,
- $v_i v_j'$ and $v_i' v_j$, for each edge $v_i v_j \in E$, and
- $v_i w_h$, $1 \leq i \leq n$, $1 \leq h \leq n-p$.

Suppose that G has a dominating set $D = \{v_1, \ldots, v_d\}$ where $d \leq p$. Note that every vertex in V' is adjacent to a vertex of D in B. We shall describe a 2-round KO-reduction scheme for B. In round 1

- for $1 \leq i \leq n$, v_i fires at v_i',
- for $1 \leq j \leq p$, v_j' fires at v_j,
- for $p+1 \leq j \leq n$, v_j' fires at a vertex in D, and
- for $1 \leq h \leq n-p$, w_h fires at a vertex in D.

Thus each vertex in $\{v_1, \ldots, v_p\}$ and V' is eliminated, and each vertex in $V \setminus \{v_1, \ldots, v_p\}$ and W survives to round 2. As the surviving vertices induce the balanced complete bipartite graph $K_{n-p,n-p}$ in B, it is clear that every surviving vertex can be eliminated in one further round.

Now suppose that B has a 2-round KO-reduction scheme. Let D be the subset of V containing vertices that are fired at in round 1. As every vertex in V' fires at — and so is adjacent to — a vertex in D, D is a dominating set in G (since each vertex in V' is joined only to copies of itself and its neighbours). We must show that $|D| \leq p$. Let $V_S = V \setminus D$ and $V'_S \subset V' \cup W$ be the sets of vertices that survive round 1. As round 2 is the final round,

$$|V_S| = |V'_S|. \tag{1}$$

As $|V' \cup W| = 2n - p$ and at most n vertices in $V' \cup W$ are fired at in round 1, $|V'_S| \geq n - p$. Thus, by (1), $|V_S| \geq n - p$. Therefore

$$\begin{aligned} |D| &= |V| - |V_S| \\ &\leq n - (n - p) \\ &= p. \end{aligned}$$

Case 2. $k = 3$. Let $J = (Q, \mathcal{S})$ be an instance of H2C. Let I' be the incidence graph of its triple $J' = (X, \mathcal{Z})$. Recall that $\mathcal{Z} = \mathcal{S}' \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{T}$. From I', we obtain a further bipartite graph G by connecting each vertex with a path as follows:

- For each vertex x in X, w add a path $H^x = y_1^x y_2^x y_3^x$ and join x to y_1^x.
- For each vertex R in $\mathcal{R}$, add a path $H^R = y_1^R \dots y_4^R$ and join R to y_1^R.
- For each vertex T in $\mathcal{T}$, add a path $H^T = y_1^T \dots y_4^T$ and join T to y_1^T.
- For each vertex P in $\mathcal{P}$, add a path $H^P = y_1^P \dots y_7^P$ and join P to the centrevertex y_4^P.
- For each vertex S' in $\mathcal{S}'$, add a path $H^{S'} = y_1^{S'} \dots y_7^{S'}$ and join S' to the centrevertex $y_4^{S'}$.

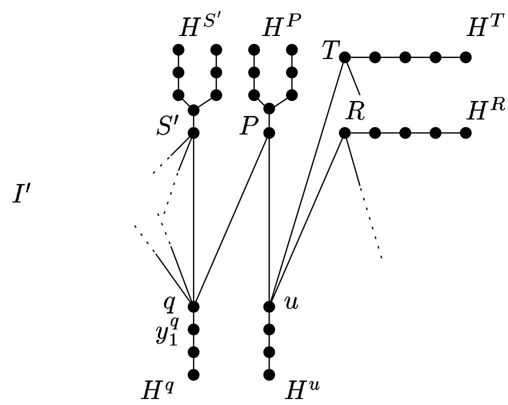

Fig. 2. The graph G in Case 2.

Figure 2 illustrates G. We shall prove that J is 2-colourable if and only if $\text{pko}(G) \leq 3$. Throughout the proof, G_1 and G_2 denote the graphs induced by the surviving vertices after, respectively, 1 and 2 rounds of a KO-reduction scheme.

Suppose $B \cup W$ is a 2-colouring of J. By Proposition 1, J' has a 2-colouring $B' \cup W'$. We define a three-round KO-reduction scheme for G.

Round 1. Vertices of degree 1 and their neighbours fire at each other. Each H^P with $P \in \mathcal{P}$ and each $H^{S'}$ with $S' \in \mathcal{S}'$ is a centred path of G, and the vertices fire as in Observation 3. For each $z \in \mathcal{R} \cup \mathcal{T}$, vertex y_1^z fires at y_2^z and y_2^z fires at y_3^z. Each vertex in $\mathcal{Z}$ fires at one of its neighbours in B'. Each vertex x in X fires at its neighbour y_1^x in H^x. Each y_1^x with $x \in B'$ fires at x. Each y_1^x with $x \in W'$ fires at y_2^x.

Thus every vertex in W' and no vertex in B' survives. Also every vertex in $\mathcal{Z}$ survives. Each vertex $z \in \mathcal{R} \cup \mathcal{T}$ is adjacent to a vertex y_1^z of degree 1, and each vertex $z \in \mathcal{S}' \cup \mathcal{P}$ is adjacent to a vertex y_4^z whose only other neighbour is a vertex y_3^z of degree 1.

Round 2. Because $B' \cup W'$ is a 2-colouring of $J = (X, \mathcal{Z})$, every vertex in $\mathcal{Z}$ has a neighbour in W' in G_1. For each $S'_j \in \mathcal{S}'$ we choose one neighbour in W' and let W'' be the set of selected vertices. Since no two vertices in $\mathcal{S}'$ have a common neighbour in X, $|W''| = n$. The vertices in G_1 fire as follows. Vertices of degree 1 and their neighbours fire at each other. Each vertex $P \in \mathcal{P}$ with a neighbour in $W' \backslash W''$ fires at this neighbour. Otherwise P fires at y_4^P. Each $x \in X$ fires at its neighbour in $\mathcal{P}$. Each $S' \in \mathcal{S}'$ fires at $y_4^{S'}$.

Thus the vertex set of G_2 is $W'' \cup \mathcal{S}'$.

Round 3. Each $S' \in \mathcal{S}'$ and its unique neighbour in W'' fire at each other, which leaves us with the null graph.

Now we suppose that $\mathrm{pko}(G) \leq 3$. We assume that a particular KO-reduction scheme for G is given and prove that J has a 2-colouring.

Claim 1. If a vertex in a set Q_i is knocked out in the first round, then all vertices in Q_i are knocked out in the first round.

Suppose that vertex $q_i^k \in Q_i$ is knocked out in the first round. We show that q_i^{k+1} (with $q_i^{\ell(i)+1} = q_i^1$) is also knocked out in the first round.

If $q_i^k \in Q_i$ is knocked out in the first round, then, by Observation 2, q_i^k fires at $y_1^{q_i^k}$. Suppose q_i^{k+1} is *not* knocked out in the first round. Observation 3 implies that P_i^{k+1} must fire at u_i^{k+1} and P_i^k must fire at either q_i^k or u_i^k. If P_i^k fires at u_i^k, then by Observation 2 u_i^k fires at $y_1^{q_i^k}$. Since vertices in $H^{P_i^k}$ must fire as in Observation 3, this means that G_1 contains a component isomorphic to a path on three vertices. By Observation 1 G_1 is not KO-reducible. Hence, P_i^k fires at q_i^k.

For the same reason R_i^{k+1} or T_i^{k+1} cannot fire at u_i^k, and consequently, fire at $y_1^{R_i^{k+1}}$ and $y_1^{T_i^{k+1}}$ respectively. Due to Observation 2 this implies that $y_1^{R_i^{k+1}}$ fires at $y_2^{R_i^{k+1}}$, and $y_1^{T_i^{k+1}}$ fires at $y_2^{T_i^{k+1}}$.

In G_1 both T_i^k and R_i^k have exactly the same neighbours, namely u_i^k and q_i^{k+1}. If T_i^k and R_i^k fire at a different neighbour in the second round, then due to Observation 2 both will be isolated vertices in G_2. Suppose T_i^k and R_i^k fire at the same neighbour. Then in all possible schemes G_2 will contain two vertices of degree 1 having the same neighbour. Observation 1 implies that G_2 is not KO-reducible. We conclude that q_i^{k+1} must be knocked out in the first round as well, and this proves the claim.

Claim 2. If a vertex in a set U_i is knocked out in the first round, then all vertices in U_i are knocked out in the first round.

This claim is proven by using the same arguments as in Claim 1.

By Claim 1 and Claim 2 we may define a set $B' \subseteq X$ as follows. All vertices of a set Q_i or U_i are in B' if and only if the set is knocked out in the first round. Let $W' = X \backslash B'$.

Claim 3. For all $1 \leq i \leq m$, either $Q_i \subseteq B'$ and $U_i \subseteq W'$, or $Q_i \subseteq W'$ and $U_i \subseteq B'$.

Let $1 \leq i \leq m$. By Observation 3, each vertex $P_i^k \in \mathcal{P}_i$ must fire at either q_i^k or u_i^k in the first round. The previous two claims imply that Q_i or U_i is knocked out in the first round. Suppose both sets are knocked out in the first round. Then, by Observation 2, u_i^1 fires at $y_1^{u_i^1}$ and q_i^1 fires at $y_1^{q_i^1}$. Then, by Observation 3, P_i^1 will not be knocked out in any round. The claim is proved.

By Claim 3, all vertices in $\mathcal{Z} \backslash \mathcal{S}'$ have one neighbour in B' and one neighbour in W'. Let S_j' be a vertex in $\mathcal{S}$. By Observation 3, S_j' fires at a neighbour in $\bigcup_{i=1}^m Q_i$. By definition, this neighbour is in B'. By both Observation 2 and Observation 3, S_j' is knocked out by a neighbour in $\bigcup_{i=1}^m Q_i$ that is not knocked out in the first round. By definition, this neighbour is in W'. It is now clear that $B' \cup W'$ is a 2-colouring of J' such that for each $1 \leq i \leq m$ either $Q_i \subseteq B'$ and $U_i \subseteq W'$, or $Q_i \subseteq W'$ and $U_i \subseteq B'$. Hence, by Proposition 1, J also has a 2-colouring.

Case 3. $k \geq 4$. We use reduction from H2C. From an instance $J = (Q, \mathcal{S})$ we construct the graph G as in the previous case. We claim that J is 2-colourable if and only if $\text{pko}(G) \leq k$.

Suppose that J is 2-colourable. As we have seen in the previous case this implies that $\text{pko}(G) \leq 3 \leq k$.

Suppose that $\text{pko}(G) \leq k$. Then G is KO-reducible. Note that in the proof of the previous case we only assume that G is KO-reducible. Hence we can copy the proof of the previous case. This completes the proof of Theorem 1. □

Corollary 1. *The* PKO *problem is* NP*-complete, even if instances are restricted to the class of bipartite graphs.*

Proof. We use reduction from H2C. From an instance $J = (Q, \mathcal{S})$ we construct the graph G as in the proof of Theorem 1. We claim that J is 2-colourable if and only if G is KO-reducible.

Suppose that J is 2-colourable. As we have seen in the proof of Theorem 1 this implies that $\text{pko}(G) \leq 3$. Hence G is KO-reducible.

Suppose that G is KO-reducible. We copy the proof of Case 2 of Theorem 1. □

EXACT PARALLEL KNOCK-OUT (k) (EPKO(k))
Instance: A graph G.
Question: Is $\mathrm{pko}(G) = k$?

Corollary 2. *The* EPKO(k) *problem is polynomially solvable for $k = 1$ and is* NP*-complete for $k \geq 2$, even if instances are restricted to the class of bipartite graphs.*

Proof. For the case $k = 1$ we only have to exclude the null graph. Let $k \geq 2$. In [1] a family of trees Y_ℓ is constructed with $\mathrm{pko}(Y_\ell) = \ell$ for $\ell \geq 1$. For the case $k = 2$ we only have to add a disjoint copy of the tree Y_2 (a path on 7 vertices) to the graph B in the proof of Case 1 in Theorem 1. For $k \geq 3$ it suffices to add a disjoint copy of the tree Y_k to the graph G constructed in the proof of Case 2 in Theorem 1. Note that the size of a tree Y_k only depends on k and not on the size of our input graph G (so we do not need the exact description of this family). □

5 Bounded tree-width

In this section we use *monadic second-order logic*; that is, that fragment of second-order logic where quantified relation symbols must have arity 1. For example, the following sentence, which expresses that a graph (whose edges are given by the binary relation E) can be 3-coloured, is a sentence of monadic second-order logic:

$$\exists R \exists W \exists B \Big\{ \forall x \Big((R(x) \vee W(x) \vee B(x)) \ \wedge \neg(R(x) \wedge W(x)) \\ \wedge \neg(R(x) \wedge B(x)) \wedge \neg(W(x) \wedge B(x)) \Big) \wedge \forall x \forall y \Big(\ E(x,y) \Rightarrow \\ (\neg(R(x) \wedge R(y)) \wedge \neg(W(x) \wedge W(y)) \wedge \neg(B(x) \wedge B(y))) \Big) \Big\}$$

(the quantified unary relation symbols are R, W and B, and should be read as sets of 'red', 'white' and 'blue' vertices, respectively). Thus, in particular, there exist **NP**-complete problems that can be defined in monadic second-order logic.

A seminal result of Courcelle [3] is that on any class of graphs of bounded tree-width, every problem definable in monadic second-order logic can be solved in time linear in the number of vertices of the graph. Moreover, Courcelle's result holds not just when graphs are given in terms of their edge relation, as in the example above, but also when the domain of a structure encoding a graph G consists of the disjoint union of the set of vertices and the set of edges, as well as unary relations V and E to distinguish the vertices and the edges, respectively, and also a binary incidence relation I which denotes when a particular vertex is incident with a particular edge (thus, $I \subseteq V \times E$). The reader is referred to [3] for more details and also for the definition of tree-width which is not required here. To prove Theorem 2, we need only prove the following proposition.

Proposition 2. *For $k \geq 1$, $\mathrm{PKO}(k)$ can be defined in monadic second order logic.*

Proof. Recall that a parallel knock-out scheme for a graph $G = (V, E)$ is a sequence of graphs

$$G \rightsquigarrow G_1 \rightsquigarrow G_2 \rightsquigarrow \cdots \rightsquigarrow G_r,$$

where G_r is the null graph. Let $W_0 = V$ and, for $1 \leq i \leq r$, let W_i be the vertex set of G_i. If we can write a formula $\Phi(W_i, W_{i+1})$ of monadic second-order logic that says

> *there exists a KO-selection f_i on W_i such that the vertex set of the KO-successor is W_{i+1},*

then we could prove the proposition with the following sentence Ω_k which is satisfied if and only if G is in $\mathrm{PKO}(k)$:

$$\begin{aligned}
&\exists W_0 \exists W_1 \cdots \exists W_k (\forall v (W_0(v) \Leftrightarrow V(v)) \\
&\wedge \Phi(W_0, W_1) \wedge \Phi(W_1, W_2) \wedge \cdots \wedge \Phi(W_{k-1}, W_k) \\
&\wedge (\forall v (\neg W_k(v) \Leftrightarrow V(v))).
\end{aligned}$$

(Here and elsewhere we have presupposed that each W_i is a set of vertices; we could easily include additional clauses to check this explicitly.)

The following claim will help us write $\Phi(W_i, W_{i+1})$.

Claim 4. There is a KO-selection f_i on W_i such that W_{i+1} is the vertex set of the KO-successor if and only if there is a partition V_1, V_2, V_3 of W_i and subsets E_1, E_2, E_3 of E such that

(a) for $j = 1, 2, 3$, each vertex in V_j is incident with exactly one edge of E_j, this edge joins it to a vertex in $W_i \setminus V_j$, and this accounts for every edge in E_j (so $|V_j| = |E_j|$).
(b) $W_{i+1} \subset W_i$ and, for $j = 1, 2, 3$, $W_{i+1} \cap V_j$ is the set of vertices in V_j not incident with edges in $E_{j'}$ for any $j' \neq j$.

We will prove the claim later. First we use it to write $\Phi(W_i, W_{i+1})$.

The following formula $\psi(V_1, E_1, V_2, E_2, V_3, E_3, W_i)$ checks that the sets V_1, V_2 and V_3 partition W_i, that the sets E_1, E_2, E_3 are edges in the graph, and that (a) is satisfied.

$$\begin{aligned}
&\forall v ((V_1(v) \vee V_2(v) \vee V_3(v)) \Leftrightarrow W_i(v)) \wedge \forall v (\neg (V_1(v) \wedge V_2(v)) \\
&\wedge \neg (V_1(v) \wedge V_3(v)) \wedge \neg (V_2(v) \wedge V_3(v))) \\
&\wedge \forall x ((E_1(x) \vee E_2(x) \vee E_3(x)) \Rightarrow E(x)) \\
&\wedge \forall x (E_1(x) \Rightarrow \exists u \exists v (V_1(u) \wedge (V_2(v) \vee V_3(v)) \wedge I(u, x) \wedge I(v, x))) \\
&\wedge \forall x (E_2(x) \Rightarrow \exists u \exists v (V_2(u) \wedge (V_1(v) \vee V_3(v)) \wedge I(u, x) \wedge I(v, x))) \\
&\wedge \forall x (E_3(x) \Rightarrow \exists u \exists v (V_3(u) \wedge (V_1(v) \vee V_2(v)) \wedge I(u, x) \wedge I(v, x))) \\
&\wedge \forall v (V_1(v) \Rightarrow \exists ! x (I(v, x) \wedge E_1(x))) \\
&\wedge \forall v (V_2(v) \Rightarrow \exists ! x (I(v, x) \wedge E_2(x))) \\
&\wedge \forall v (V_3(v) \Rightarrow \exists ! x (I(v, x) \wedge E_3(x)))
\end{aligned}$$

(The semantics of $\exists!$ is 'there exists exactly one'; clearly, this abbreviates a more complex though routine first-order formula.) The following formula checks that (b) is satisfied and is denoted $\chi(V_1, E_1, V_2, E_2, V_3, E_3, W_i, W_{i+1})$.

$$\begin{aligned}\forall v(W_{i+1}(v) \Leftrightarrow (W_i(v) \wedge ((V_1(v) \wedge \neg\exists x((E_2(x) \vee E_3(x)) \wedge I(v,x)))\\ \vee(V_2(v) \wedge \neg\exists x((E_1(x) \vee E_3(x)) \wedge I(v,x)))\\ \vee(V_3(v) \wedge \neg\exists x((E_1(x) \vee E_2(x)) \wedge I(v,x)))).\end{aligned}$$

And now we can write $\Phi(W_i, W_{i+1})$:

$$\begin{aligned}\exists V_1 \exists E_1 \exists V_2 \exists E_2 \exists V_3 \exists E_3 (\psi(V_1, E_1, V_2, E_2, V_3, E_3, W_i)\\ \wedge\chi(V_1, E_1, V_2, E_2, V_3, E_3, W_i, W_{i+1})).\end{aligned}$$

It only remains to prove Claim 4. Suppose that we have sets V_1, V_2, V_3, E_1, E_2 and E_3 that satisfy the conditions of the claim. Then to define the KO-selection f_i, for $j = 1, 2, 3$, for each vertex $v \in V_j$, let v fire at the unique neighbour joined to v by an edge in E_j. It is easy to check that W_{i+1} is the vertex set of the KO-successor.

Now suppose that we have a KO-selection f_i. Let H_i be the spanning subgraph of G_i with edge set $\{vf_i(v) \mid v \in W_i\}$. The firing can be represented as an orientation of H: orient each edge from v to $f_i(v)$ (some edges may be oriented in both directions). As each vertex has exactly one edge oriented away from it, each component of the oriented graph contains one directed cycle, of length at least 2, with a pendant in-tree attached to each vertex of the cycle; see Figure 3.

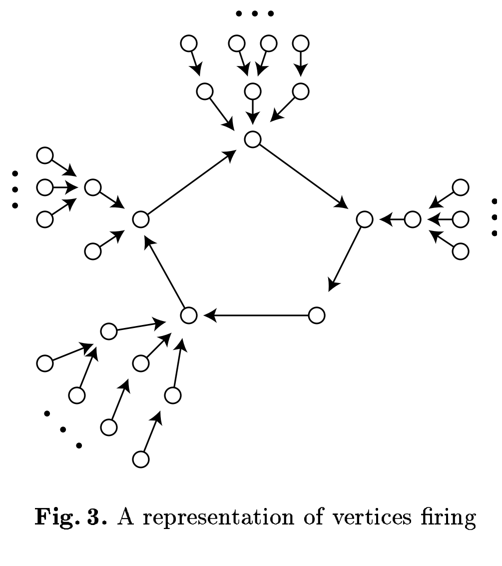

Fig. 3. A representation of vertices firing

We find the sets $V_1, V_2, V_3, E_1, E_2, E_3$; the edge sets contain only edges of H_i. We may assume that H_i is connected (else we can find the sets componentwise).

Let the vertices of the unique cycle in the orientation be $v_1, \ldots, v_c$ where the edges are $v_l v_{l+1}$, $1 \le l \le c-1$, and $v_c v_1$. So H_i contains vertices $v_1, \ldots, v_c$ with a pendant tree (possibly trivial) attached to each.

For $1 \le l \le c$, let U_e^l be the set of vertices in the pendant tree attached to v_l whose distance from v_l is even (but not zero), and let U_o^l be the vertices in the tree at odd distance from v_l. Let

$$\begin{aligned} V_1 &= \bigcup_{l \text{ odd}} U_o^l \;\cup\; \bigcup_{l \text{ even}} U_e^l \;\cup \{v_l : l \text{ is even}, l \neq c\}, \\ V_2 &= \bigcup_{l \text{ odd}} U_e^l \;\cup\; \bigcup_{l \text{ even}} U_o^l \;\cup \{v_l : l \text{ is odd}, l \neq c\}, \text{ and} \\ V_3 &= \{v_c\}, \end{aligned}$$

and, for $i = 1, 2, 3$, let E_i contain $vf_i(v)$ for each $v \in V_i$. It is clear that the sets we have chosen satisfy the conditions of the claim.

This completes the proof of the claim and of the proposition. □

Theorem 2 follows from the proposition. And, noting that EPKO(k) is defined by the monadic second-order sentence $\Omega_k \wedge \neg \Omega_{k-1}$, we have the following result.

Corollary 3. *For $k \geq 1$,* EPKO*(k) is solvable in linear time on any class of graphs with bounded tree-width.*

Finally, we note that to check whether a graph G is reducible it is sufficient to check whether $\text{pko}(G) = k$, for $1 \le k \le \Delta$, where Δ is the maximum degree of G. Thus G is reducible if and only if the sentence $\Omega_\Delta \vee \Omega_{\Delta-1} \vee \cdots \vee \Omega_1$ is satisfied. This gives us our last result.

Corollary 4. *On any class of graphs with bounded tree-width,* PKO *can be solved in polynomial time.*

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