Amalgamations of factorizations of complete equipartite graphs

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Abstract

Let t be a positive integer, and let $L = (l_1, \ldots, l_t)$ and $K = (k_1, \ldots, k_t)$ be collections of nonnegative integers. A graph has a (t, K, L) factorization if it can be represented as the edge-disjoint union of factors $F_1, \ldots, F_t$ where, for $1 \leq i \leq t$, F_i is k_i-regular and at least l_i-edge-connected. In this paper we consider (t, K, L)-factorizations of complete equipartite graphs. First we show precisely when they exist. Then we solve two embedding problems: we show when a factorization of a complete σ-partite graph can be embedded in a (t, K, L)-factorization of a complete s-partite graph, $\sigma < s$, and also when a factorization of $K_{a,b}$ can be embedded in a (t, K, L)-factorization of $K_{n,n}$, $a, b \leq n$. Our proofs use the technique of amalgamations of graphs.

1 Introduction

We denote the complete s-partite graph with n vertices in each part $K_n^{(s)}$. Let t be a positive integer, let $K = (k_1, k_2, \ldots, k_t)$ and $L = (l_1, l_2, \ldots, l_t)$ where, for $1 \leq i \leq t$, k_i is a positive integer and l_i is a nonnegative integer. A factorization $F_1, \ldots, F_t$ of a graph such that, for $1 \leq i \leq t$, F_i is k_i-regular and has edge-connectivity at least l_i is called a (t, K, L)-factorization. We describe exactly when $K_n^{(s)}$ has a (t, K, L)-factorization:

Theorem 1 *A (t, K, L)-factorization of $K_n^{(s)}$ exists if and only if*

(A1) $\displaystyle\sum_{i=1}^{t} k_i = n(s-1)$,

(A2) *if ns is odd then each k_i is even,*

(A3) *for $1 \leq i \leq t$, $l_i \leq k_i$, and*

(A4) *$l_i = 0$ if $k_i = 1$.*

If $n \in \{1, 2\}$, then $K_n^{(s)}$ is either the complete graph or the complete graph less a one-factor. These cases of Theorem 1 were first proved by Johnstone [6]. They were subsequently proved by Johnson [5] using amalgamations, and here we attempt to generalize the results and techniques of that paper (which presented many results on (t, K, L)-factorizations of complete graphs) to obtain results on complete equipartite graphs.

We believe that the only other non-trivial case of Theorem 1 previously proved is when each $k_i = l_i = 2$, that is when each factor is a Hamilton cycle. This case was first proved by Laskar and Auerbach [7], who constructed the factorizations, and independently by Hilton and Rodger [4] using amalgamations.

We sketch how the technique of amalgamations is used. This will lead us to the other theme of this paper: embeddings.

1.1 Amalgamations

Consider a partition of a graph G's vertex set into subsets $V_1, \dots, V_r$. Then an amalgamation of G has vertex set $V_1, \dots, V_r$ and for each edge in G joining a pair of vertices in V_i, $1 \leq i \leq r$, there is a loop on V_i in the amalgamation, and for each edge in G joining a vertex in V_i to a vertex in V_j, $1 \leq i < j \leq r$, there is an edge V_iV_j in the amalgamation. (We can think of the amalgamation as being obtained from G by merging vertices that belong to the same subset whilst retaining all edges.)

If G has a factorization, then we can represent it as an edge-colouring: the factors are the colour classes (in this paper we frequently use the equivalence of factorizations and edge-colourings). This colouring can be transferred to an amalgamation of G—each edge of the amalgamation has the same colour as the corresponding edge of G. In what follows when we refer to an amalgamation we mean a graph that has been edge-coloured. Suppose that $G = K_n^{(s)}$ and that it has a particular type of factorization, say a Hamiltonian decomposition. Then we can find some properties that an amalgamation of G must possess. For example we can find the number of loops on each vertex, the number of edges between each pair of vertices and the number of edges of each colour incident with each vertex. We call *any* edge-coloured graph that satisfies these properties an outline Hamiltonian decomposition of $K_n^{(s)}$. The aim when using amalgamations is to prove that every outline graph is an amalgamated graph. So in our example, for each outline Hamiltonian decomposition we would have to find a Hamiltonian decomposition of which it is an amalgamation.

1.2 Embeddings

Amalgamations can be used to prove embedding results. Suppose that we have a factorization (or an edge-colouring) of $K_n^{(\sigma)}$. Add to it a vertex v. Join v to each vertex of $K_n^{(\sigma)}$ by $n(s-\sigma)$ edges and put $n^2\binom{s-\sigma}{2}$ loops on v to form a graph G. Complete the edge-colouring of G by colouring the edges incident with v. (Note that G can be seen to be $K_n^{(s)}$ with $n(s-\sigma)$ vertices merged.) If G is an outline factorization (of some specified type) of $K_n^{(s)}$ and we have proved that every outline graph is an amalgamated graph, then there is factorization of $K_n^{(s)}$ in which the factorization of $K_n^{(\sigma)}$ is embedded; we can think of this factorization of $K_n^{(s)}$ as being obtained from G by splitting v into $n(s-\sigma)$ vertices. From the properties that define an outline factorization we can work back to find the properties that the factorization of $K_n^{(\sigma)}$ must possess if it is to be embedded.

Hilton [1] first used the technique of amalgamations in the context of embedding factorizations of graphs: he considered Hamiltonian decompositions of the complete graph. Generalizations of his results to decompositions of the complete graph into regular factors of prescribed degree and edge-connectivity have been proved by various authors; see, for example, [3, 8, 10]. The most general result of this kind was obtained by Johnson [5] who considered (t, K, L)-factorizations of the complete graph. Hilton, with Rodger, generalized his original result in a different direction by considering Hamiltonian decompositions of the complete equipartite graph [4]. In this paper, we unite these two strands of research by considering (t, K, L)-factorizations of the complete equipartite graph.

In the next section we formally introduce amalgamations of (t, K, L)-factorizations of complete equipartite graphs, and at the end of the section we use amalgamations to prove Theorem 1. In the final section we consider embedding problems. We suppose that we have a factorization of $K_n^{(\sigma)}$, and ask when it can be embedded in a (t, K, L)-factorization of $K_n^{(s)}$, $\sigma < s$. We also look at embedding factorizations of $K_{a,b}$ in (t, K, L)-factorizations of $K_{n,n}$, $a, b \leq n$.

As noted before, $K_1^{(s)} = K_s$ and $K_2^{(s)} = K_{2s} - I$, where I is a 1-factor. Results on amalgamations and embeddings of (t, K, L)-factorizations of these graphs are already known and can be found in [5]. So in this paper we assume throughout that $n \geq 3$.

2 Amalgamated factorizations

2.1 Detachments

Before we formally define amalgamations we require another definition. Let D and G be graphs. D is a *detachment* of G if there is a bijection ρ: $E(D) \longrightarrow E(G)$ and a surjection σ: $V(D) \longrightarrow V(G)$ such that

- if e is a loop on v in D, then $\rho(e)$ is a loop on $\sigma(v)$ in G,
- if e is an edge joining v and w in D and $\sigma(v) = \sigma(w)$, then $\rho(e)$ is a loop on $\sigma(v)$ in G, and
- if e is an edge joining v and w in D and $\sigma(v) \neq \sigma(w)$, then $\rho(e)$ is an edge joining $\sigma(v)$ and $\sigma(w)$ in G.

We can think of D as being obtained from G by splitting vertices. Some authors refer to detachments as *disentanglements*.

Let G be a graph of which we seek to find a detachment. We define three functions f, c, e: $\mathcal{P}(V(G)) \longrightarrow \mathbf{Z}$, ($\mathcal{P}(V(G))$ is the power set of $V(G)$). For each set of vertices $V \subseteq V(G)$, let $f(V)$ be the total number of vertices we wish to split the vertices of V into, let $c(V)$ be the number of components in $G-V$, and let $e(V)$ be the number of edges (including loops) that are incident with at least one vertex in V (loops and edges incident twice with vertices in V are only counted once). We need the following result of Nash-Williams [9].

Proposition 2 *Let k and l be nonnegative integers. Let G be a graph (possibly containing multiple edges and loops) in which the degree of each vertex is a multiple of k. Then G has an l-edge-connected k-regular detachment if and only if*

(X1) *G is l-edge-connected,*

(X2) *if $l = 1$, then for all $V \subseteq V(G)$, $f(V) + c(V) \leq e(V) + 1$,*

(X3) *if l is odd and $l = k$, then G has no cutvertex with degree $2l$, and*

(X4) *if l is odd and $l = k$, then G is not a loopless graph that contains exactly two vertices each with degree $2l$.* □

2.2 Amalgamations

An amalgamation is the opposite of a detachment, except that we define amalgamations on graphs which have an edge-colouring. Let t be a positive integer. Let F and H be t-edge-coloured graphs. H is an *amalgamation* of F if there is a bijection ϕ: $E(F) \longrightarrow E(H)$ and a surjection ψ: $V(F) \longrightarrow V(H)$ such that

- if e is a loop coloured i on v in F, then $\phi(e)$ is a loop coloured i on $\psi(v)$ in H,
- if e is an edge coloured i joining v and w in F and $\psi(v) = \psi(w)$, then $\phi(e)$ is a loop coloured i on $\psi(v)$ in H, and
- if e is an edge coloured i joining v and w in F and $\psi(v) \neq \psi(w)$, then $\phi(e)$ is an edge coloured i joining $\psi(v)$ and $\psi(w)$ in H.

Let F_i and H_i be the subgraphs of F and H induced by edges coloured i, $1 \leq i \leq t$.

Let t, n, K and L satisfy conditions (A1) to (A4) of Theorem 1. Suppose that $F = K_n^{(s)}$ is t-edge-coloured and that F_i is k_i-regular and l_i-edge-connected, $1 \leq i \leq t$. We think of the vertex set of $K_n^{(s)}$ as being composed of s parts $P_1, \ldots, P_s$ where each part is a set of n independent vertices. If H is an amalgamation of F, then define f: $V(H) \longrightarrow \mathbf{N}$ by

$$f(v) = |\{u : u \in V(K_n^{(s)}), \psi(u) = v\}|,$$

and, for $1 \leq h \leq s$, define f_h: $V(H) \longrightarrow \mathbf{N}$ by

$$f_h(v) = |\{u : u \in P_h, \psi(u) = v\}|.$$

So f counts the vertices that are merged to form v and, for $1 \leq h \leq s$, f_h tells us how many of these vertices are from P_h. Together H, f and f_h, $1 \leq h \leq s$, form an *amalgamated* (t, K, L)*-factorization of* $K_n^{(s)}$.

Proposition 3 *Let H, f and f_h, $1 \leq h \leq s$, be an amalgamated (t, K, L)-factorization of $K_n^{(s)}$. Then*

(B1) *for all pairs of distinct vertices $v, w \in V(H)$, there are* $\displaystyle\sum_{\substack{h_1,h_2 \in \{1,\ldots,s\} \\ h_1 \neq h_2}} f_{h_1}(v) f_{h_2}(w)$ *edges joining v to w,* (B2) *for all* $v \in V(H)$*, there are* $\displaystyle\sum_{1 \le h_1 < h_2 \le s} f_{h_1}(v) f_{h_2}(v)$ *loops on* v*,*

(B3) *for all* $v \in V(H)$*, for* $1 \le i \le t$*,* v *is incident with* $k_i f(v)$ *edges of colour* i *(counting loops twice),*

(B4) $\displaystyle\sum_{v \in V(H)} f(v) = ns$*, and, for* $1 \le h \le s$*,* $\displaystyle\sum_{v \in V(H)} f_h(v) = n$*, and*

(B5) *for* $1 \le i \le t$*,* H_i *has an* l_i*-edge-connected* k_i*-regular detachment.*

Proof: The number of edges joining vertices v and w (possibly $v = w$) in the amalgamation is equal to the number of edges in $K_n^{(s)}$ joining a vertex merged to form v to a vertex merged to form w, and pairs of vertices are joined by one edge in $K_n^{(s)}$ unless they are in the same part. This is enough to prove (B1) and (B2). There are $f(v)$ vertices merged to form v and each is incident with k_i edges coloured i, $1 \le i \le t$, so (B3) is satisfied. As we noted f and f_h count vertices in $V(K_n^{(s)})$ and P_h respectively. In each case each vertex in the set is counted exactly once so (B4) is satisfied. Finally, for (B5), note that F_i is a l_i-edge-connected k_i-regular detachment of H_i. □

2.3 Outline factorizations

A t-edge-coloured graph H, a function f: $V(H) \longrightarrow \mathbf{N}$ and functions f_h: $V(H) \longrightarrow \mathbf{N}$, $1 \le h \le s$, form an *outline* (t, K, L)*-factorization of* $K_n^{(s)}$ if they satisfy (B1) to (B5). By Proposition 3, an amalgamated (t, K, L)-factorization of $K_n^{(s)}$ is an outline (t, K, L)-factorization of $K_n^{(s)}$. As we shall see, the converse is not true in general. However, we can prove that a particular type of outline factorization of $K_n^{(s)}$ is an amalgamated factorization.

Theorem 4 *Let* H*,* f *and* f_h*,* $1 \le h \le s$*, be an outline* (t, K, L)*-factorization of* $K_n^{(s)}$ *such that* $l_i \ne 1$*,* $1 \le i \le t$*. Then* H*,* f *and* f_h*,* $1 \le h \le s$ *are an amalgamated* (t, K, L)*-factorization of* $K_n^{(s)}$ *if for each* $v \in V(H)$ *either*

(Z1) *for* $1 \le h \le s$*,* $f_h(v) \in \{0, n\}$*, or*

(Z2) $f_h(v) = 0$ *for all but one value of* h*.*

Before the proof is given we make some remarks about the possibility of proving a more general outline/amalgamation theorem.

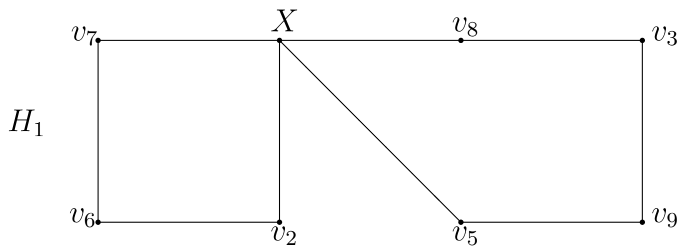
There are two restrictions on the outline factorizations covered by Theorem 4. First we have that $l_i \neq 1$, $1 \leq i \leq t$. We cannot find an example that shows that a theorem without this condition is not true, but we cannot prove such a theorem. We shall see later why we would have difficulty proving the theorem if we allowed $l_i = 1$

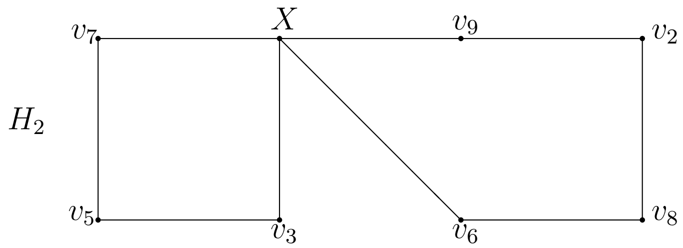
The second restriction is given by (Z1) and (Z2). Let H be an outline graph that satisfies these two conditions. Suppose that there is a factorization of $K_n^{(s)}$ of which H is an amalgamation: we can think of it as being obtained by splitting the vertices of H. (Z1) and (Z2) say that each vertex in H must be split either into vertices that comprise all of some number of the parts of $K_n^{(s)}$ or into vertices that all belong to the same part of $K_n^{(s)}$. We consider some examples that show why we impose such restrictions.

In the first example let H, f and f_h, $1 \leq h \leq 3$ be the outline $(3, K, L)$-factorization of $K_3^{(3)}$, with $K = L = (2, 2, 2)$, shown in Figure 1 (for $v_i \neq X$, $f(v_i) = 1$, $f_h(v_i) = 1$ if $\left\lceil \frac{i}{3} \right\rceil = h$, $f_h(v_i) = 0$ otherwise; $f(X) = 2$, $f_1(X) = f_2(X) = 1$, $f_3(X) = 0$). We call this an outline Hamiltonian decomposition. It is easy to check that (B1) to (B5) are satisfied, yet we can show that H, f and f_h, $1 \leq h \leq 3$ are not an amalgamation of a $(3, K, L)$-factorization of $K_3^{(3)}$. Suppose that $K_3^{(3)}$ has a Hamiltonian decomposition F_1, F_2, F_3 such that F_i is a detachment of H_i, $1 \leq i \leq 3$. Suppose also that the two vertices into which X is split are labelled v_1 and v_4 so that the parts of $K_3^{(3)}$ are $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$. Consider F_1, a 9-cycle obtained from H_1 by splitting X into two vertices, v_1 and v_4. Clearly each is adjacent to one of v_2 and v_7, and one of v_5 and v_8. The edge v_1v_2 is not in $K_3^{(3)}$ so we must have $v_1v_7 \in E(F_1)$. But by a similar argument we must also have $v_1v_7 \in E(F_2)$, a contradiction.

We have established that a general outline/amalgamation theorem cannot be proved without some restrictions. Is it possible though to lessen the restrictions of Theorem 4? In [4] Hilton and Rodger considered outline Hamiltonian decompositions of $K_n^{(s)}$. They stated that H, f and f_h, $1 \leq h \leq s$ were amalgamations of Hamiltonian decompositions if they satisfied

(Z1*) for some vertex $u \in V(H)$, $f_h(u) \in \{0, n\}$ for all but at most one value of h, and





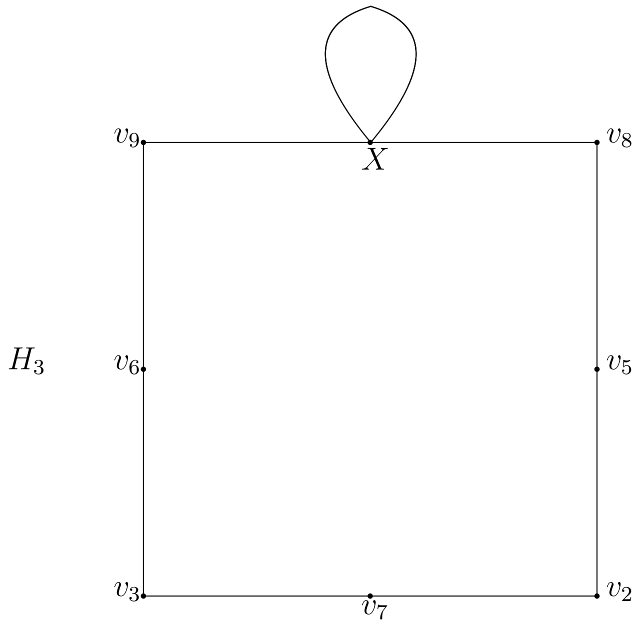


Figure 1: Outline Hamiltonian decomposition of $K_3^{(3)}$

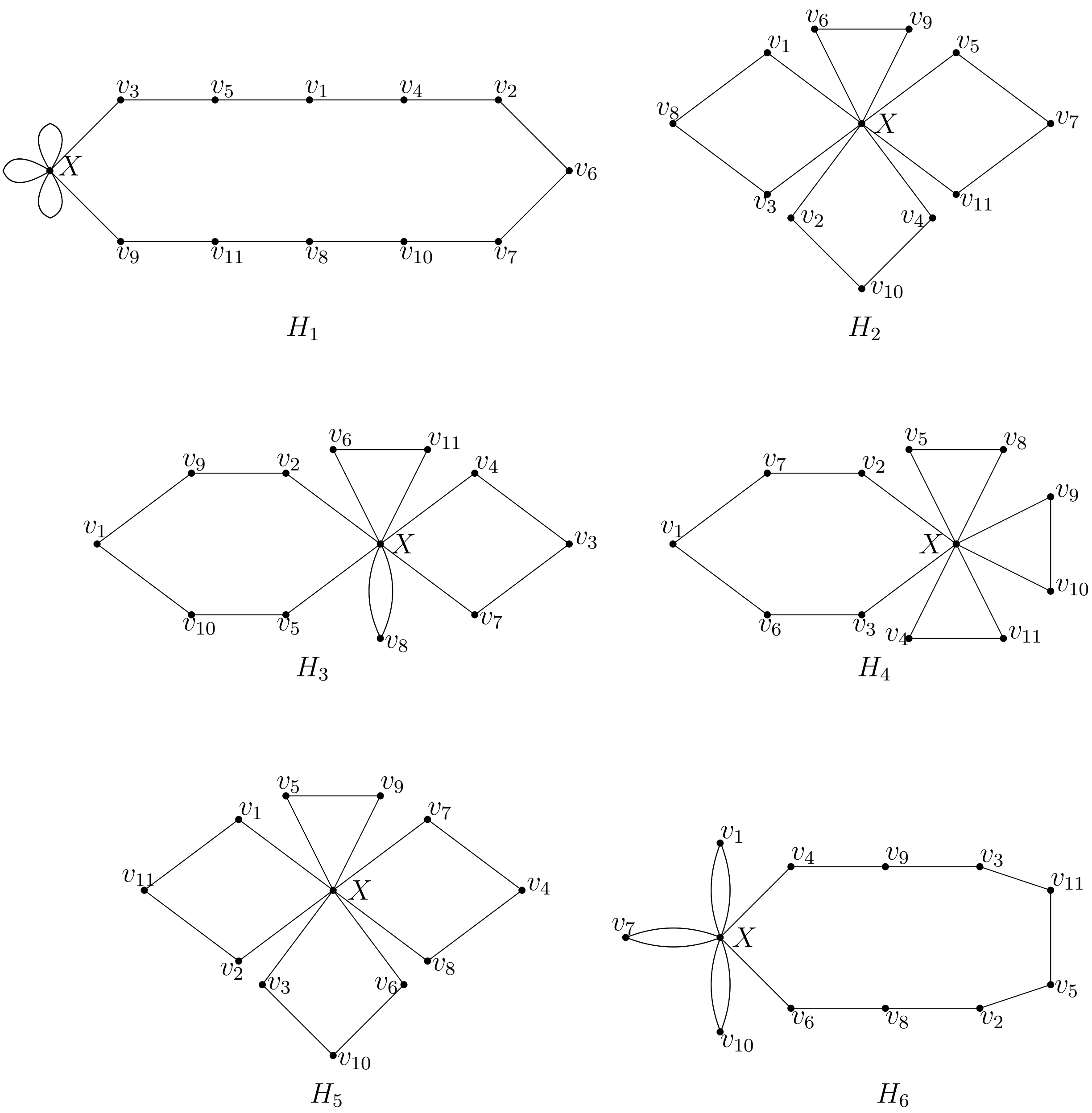


Figure 2: Outline Hamiltonian decomposition of $K_3^{(5)}$

(Z2*) for each vertex $v \in V(H) \setminus \{u\}$, $f_h(v) = 0$ for all but one value of h.

We show that this is not true. Let H, f and f_h, $1 \leq h \leq 5$ be the outline Hamiltonian decomposition of $K_3^{(5)}$ illustrated in Figure 2 (for $v_i \neq X$, $f(v_i) = 1$ and $f_h(v_i) = 1$ if $\left\lceil \frac{i}{3} \right\rceil = h$, $f_h(v_i) = 0$ otherwise; $f(X) = 4$, $f_h(X) = 0$, $1 \leq h \leq 3$, $f_4(X) = 1$, $f_5(X) = 3$). We show that h, f and f_h, $1 \leq h \leq 5$ are not an amalgamation of a Hamiltonian decomposition of $K_n^{(s)}$. Suppose there is such a decomposition into Hamilton cycles $F_1, \ldots, F_6$ and X is split into vertices labelled $v_{12}, v_{13}, v_{14}, v_{15}$ so that the parts of $K_3^{(5)}$ are $\{v_{3i+1}, v_{3i+2}, v_{3i+3}\}$, $0 \leq i \leq 4$. Therefore any loop on X in H_i corresponds to an edge in F_i joining v_{12} to one of v_{13}, v_{14}, v_{15} (since these latter three vertices are independent). But there are three loops on X in H_1 so in F_1 v_{12} must have degree at least 3, a contradiction.

We could avoid such counterexamples by extending the definition of outline factorizations. Consider a (t, K, L)-factorization of $K_n^{(s)}$, and a subset of the vertices that contains f_1 vertices from the first part and f_2 vertices from the second part. The number of edges in the subgraph of a k_i-factor induced by these vertices is at most $(\min\{f_1, f_2\} \min\{k_i, \max\{f_1, f_2\}\})$ (suppose $f_1 < f_2$; every edge in the subgraph is incident with one of the f_1 vertices in the first part, and each of these vertices has degree not more than k_i—since this is its degree in the k_i-factor—and not more than f_2—since it is joined by at most one edge to each of the f_2 vertices in the second part). Hence we can add to Proposition 3 a sixth property of amalgamated (t, K, L)-factorizations.

(B6) Each vertex v has at most

$$\sum_{1 \leq h_1 < h_2 \leq s} \min\{f_{h_1}(v), f_{h_2}(v)\} \min\{k_i, \max\{f_{h_1}(v), f_{h_2}(v)\}\}$$

loops of colour i, $1 \leq i \leq t$.

Then we could add (B6) to the definition of an outline (t, K, L)-factorization of $K_n^{(s)}$. It is possible, but not obvious, that with this extra condition Hilton and Rodger's theorem on Hamiltonian decompositions could be proved. However, in the more general case it is possible to find outline factorizations that satisfy (Z1*), (Z2*) and (B6) but are not amalgamations. We have an example, but it is too large to describe here.

2.4 Swap-sets

Before we prove Theorem 4, we must introduce an important tool first used in [2]. Let a and b be vertices each of degree d in a multigraph G. Let u be a neighbour of a and v be a neighbour of b in G. To (a,b)-*swap* the vertices u and v means to form a new graph from G by deleting the edges au and bv, and adding the edges av and bu. Clearly this manoeuvre leaves the degrees of all the vertices unaltered.

We can find d neighbours of a in G by counting a vertex u as a neighbour of a as many times as there are edges au. An (a,b)-*swap-set* is a collection of d pairs of vertices such that each neighbour of a is the first element of exactly one pair and each neighbour of b is the second element of exactly one pair. We call the pairs (a,b)-*pairs*. The proof of the following lemma uses an argument from [2]

Lemma 5 *If a and b are vertices each of degree d in a l-edge-connected multigraph G, then there exists an (a,b)-swap-set S such that a graph obtained from G by (a,b)-swapping any number of (a,b)-pairs in the swap-set is at least l-edge-connected.*

We call a swap-set that satisfies this lemma an (a,b,l)-swap-set.

Proof: First form S. In G we can find l edge-disjoint a–b paths $au_j \cdots v_j b$, $1 \leq j \leq l$. Let (u_j, v_j) be a pair in S. For any edges ab in G not already considered as one of the paths, let (b,a) be a pair in S. Complete S by pairing off the remaining neighbours of a and b arbitrarily.

Consider a graph obtained from G by (a,b)-swapping pairs in S. It contains l edge-disjoint a–b paths since, for $1 \leq j \leq l$, it contains either $au_j \cdots v_j b$ or $bu_j \cdots v_j a$. Now we use induction to prove the lemma. We know that G is l-edge-connected. Suppose that after some number of (a,b)-swaps we have obtained a graph H that is l-edge-connected, and then we (a,b)-swap a further (a,b)-pair (u,v) to obtain a graph J. That is, au and bv are deleted in H and replaced by av and bu to obtain J. If J is not l-edge connected, then we can find a minimal edge-cutset E such that $|E| < l$. We show that H has an edge-cutset of the same size as E, a contradiction. Let C_1 and C_2 be the two connected components of $J - E$. In J there are l edge-disjoint a–b paths so a and b must be in the same component of $J - E$, say C_1. If u and v are also both in C_1, then in $J - E$ we could reverse the

(a, b)-swap of u and v to obtain $H - E$ which would also have two components. If u and v are both in C_2, then av and bu must both be in E. Thus $(E \setminus \{av, bu\}) \cup \{au, bv\}$ is an edge-cutset of H. Finally, suppose that u is in C_1 and v is in C_2. Then $av \in E$ and $bu \in C_1$. Let $E' = (E \setminus \{av\}) \cup \{bv\}$ and $C_1' = (C_1 - \{bu\}) \cup \{au\}$. Thus $H - E'$ has two connected components, C_1' and C_2. □

2.5 Proof of Theorem 4

We will find a (t, K, L)-factorization of $K_n^{(s)}$ of which H, f and f_h, $1 \leq h \leq s$ are an amalgamation.

By (B5), for $1 \leq i \leq t$, H_i has an l_i-edge-connected k_i-regular detachment F_i. H_i is called a colour class and F_i is called a factor. Let $V(K_n^{(s)})$ be the vertex set of each factor. Label the vertices of each factor so that for each vertex v in H the set of vertices into which v is split when F_i is obtained from H_i is the same for each i, $1 \leq i \leq t$. Also let the number of vertices in P_h, $1 \leq h \leq s$, formed when v is split be $f_h(v)$. Let U also be a graph with vertex set $V(K_n^{(s)})$ that contains each edge of each factor. We need to alter the factors until $U = K_n^{(s)}$ whilst retaining the property that each factor F_i is a k_i-regular l_i-edge-connected detachment of the corresponding colour class H_i, $1 \leq i \leq t$.

Let $V(H) = \{v_1, v_2, \ldots, v_r\}$. Let $V(K_n^{(s)}) = V_1 \cup V_2 \cup \cdots \cup V_r$, where V_j, $1 \leq j \leq r$, is the set of vertices—called a *set of split vertices*—that was formed by the splitting of the vertex v_j in each H_i. For $1 \leq j \leq r$, $1 \leq h \leq s$, let $I_{jh} = V_j \cap P_h$; we call these sets *independent sets of split vertices.* So each set of split vertices can be partitioned into independent sets of split vertices. Note that $|I_{jh}| = f_h(v_j)$. If a part P_h of $V(K_n^{(s)})$ is a subset of a set of split vertices, then it is called a *single* part (i.e $f_h(v_j) = n$ for some j); if it contains vertices from more than one set of split vertices, then it is called a *mixed* part.

Let x and y each be either a vertex, an independent set of split vertices or a set of split vertices. Then $p(x, y)$ is the number of edges in U that join x to y and $q(x, y)$ is the number of edges in $K_n^{(s)}$ that join x to y. If $p(x, y) = q(x, y)$, then we may say that x and y are joined the correct number of times.

There are four main stages to the proof. At each stage we use (a, b)-swaps to make alterations to the factors and thus also to U. (Note that to avoid introducing further notation we use the same names—F_i, $1 \leq i \leq t$, and U—for graphs before and after making (a, b)-swaps). In (C1) to (C4) we state the property U has at the end of each stage.

(C1) For each independent set of split vertices I_{jh} and each set of split vertices V_z, $p(I_{jh}, V_z) = q(I_{jh}, V_z)$.

(C2) For each pair of independent sets of split vertices I_{jh} and I_{zg}, $p(I_{jh}, I_{zg}) = q(I_{jh}, I_{zg})$.

(C3) For each vertex v and each independent set of split vertices I_{jh}, $p(v, I_{jh}) = q(v, I_{jh})$.

(C4) For each pair of vertices v and w, $p(v, w) = q(v, w)$.

Note that when (C4) is satisfied, $U = K_n^{(s)}$ and the proof is complete.

For the first two stages we will work not with the factors F_i but with graphs F_i^* that are amalgamations of the factors and detachments of the colour classes. They are called *partially amalgamated factors* and are obtained from the factors by merging vertices that belong to the same single part. That is, they have vertex set $A \cup B$ where

$$\begin{aligned} A &= \{v \in K_n^{(s)} : v \text{ is in a mixed part}\}, \\ B &= \{P^* : P \text{ is a single part}\}, \end{aligned}$$

and for each edge uv in F_i

- if u and v are both in mixed parts, then there is an edge uv in F_i^*,
- if u is in a mixed part and v is a single part P, then there is an edge uP^* in F_i^*, and
- if u is in a single part P_1 and v is in a single part P_2, then there is an edge $P_1^* P_2^*$ in F_i^*.

Note that F_i^*, $1 \leq i \leq t$, is l_i-edge-connected. Let U^* be a graph also with vertex set $A \cup B$ that contains each edge of each partially amalgamated factor. If $V \subseteq K_n^{(s)}$ is a set of split vertices, then the subset of $A \cup B$ that comprises the vertices formed when the vertices of V were merged is also called a set of split vertices and is denoted V^*; independent subsets of split vertices in $A \cup B$ are similarly defined and denoted. Note that, by (Z1) and (Z2), in

U^* sets of split vertices contain either vertices in A or vertices in B but not both, and each vertex in B is an independent set of split vertices.

Let x and y each be either a vertex, an independent set of split vertices or a set of split vertices in U^*. Then $p^*(x,y)$ denotes the number of edges that join x to y in U^* and $q^*(x,y)$ denotes the number of edges that join x to y in an amalgamation of $K_n^{(s)}$ with vertex set $A \cup B$. Note that (B1) and (B2) say that each pair of sets of split vertices in U and U^* are joined the correct number of times.

Before we come to the four main stages of the proof, we remove any loops from the partially amalgamated factors. Note that the vertices of A belong to sets of split vertices that belong to mixed parts, and therefore, by (Z1) and (Z2), to sets of split vertices V_j^* such that $f_h(v_j) = 0$ for all but one value of h. Hence, by (B2), the vertices of A do not have any loops. Suppose that there is a loop on $P^* \in B$ in F_i^*. Let V_z^* be the set of split vertices that contains P^*. By (B2), $f_h(v_z) > 0$ for more than one value of h so there is a vertex $Q^* \in V_z^*$, $P^* \neq Q^*$. If there is also a loop on Q^*, then we delete the two loops and add two edges that each join P^* to Q^*. Otherwise we can find an edge Q^*u, $u \neq P^*$, and we delete this edge and the loop on P^* and add edges P^*Q^* and P^*u. In each case F_i^* remains an l_i-edge-connected detachment of H_i and the vertices' degrees do not change.

Let P^* and Q^* be vertices in B that belong to the same set of split vertices. Each has degree k_in in F_i^*, $1 \leq i \leq t$, and therefore each has k_in neighbours. By Lemma 5 we can find a (P^*, Q^*, l_i)-swap set. We call this set $S_i^*(P^*, Q^*)$. Recall that this is a collection of k_in (P^*, Q^*)-pairs in F_i^* such that each neighbour of P^* is the first element of exactly one pair and each neighbour of Q^* is the second element of exactly one pair and that if we (P^*, Q^*)-swap pairs in $S_i^*(P^*, Q^*)$, then F_i^* remains l_i-edge-connected, and as P^* and Q^* belong to the same set of split vertices, F_i^* remains a detachment of H_i.

We show that after performing any number of (P^*, Q^*)-swaps on F_i^*, we can always find a detachment F_i that is an l_i-edge-connected k_i-factor of $K_n^{(s)}$. Proposition 2 tells us when it is possible to find such detachments. Of the four conditions, (X2) does not apply since we have that $l_i \neq 1$, $1 \leq i \leq t$, and (X3) and (X4) do not apply since $n \neq 2$ so F_i^* has no vertex of degree $2k_i$. Thus we only require that (X1) is satisfied, and as we have just noted, F_i^* remains l_i-edge-connected. (We observe that if $l_i = 1$, then (X2) would not necessarily remain satisfied after a (P^*, Q^*)-swap. This is the reason that we cannot prove the theorem if we allow $l_i = 1$.)

We recast (C1) and (C2) in terms of the partially amalgamated factors. Consider (C1). Each independent set of split vertices in A is also a set of split vertices so by (B1) is already joined the correct number of times to every other set of split vertices. We must alter the partially amalgamated factors so that each independent set of split vertices in B is joined the correct number of times to each set of split vertices. But the independent sets of split vertices in B are its vertices so we require that

(C1*) for each $P^* \in B$, for $1 \leq j \leq r$, $p^*(P^*, V_j^*) = q^*(P^*, V_j^*)$.

When (C1*) is satisfied each independent set of split vertices in A will be joined the correct number of times to every other independent set of split vertices (in A and B). We require that the same is true for independent sets of split vertices in B so we further alter the partially amalgamated factors so that

(C2*) for each distinct pair $P^*, Q^* \in B$, $p^*(P^*, Q^*) = q^*(P^*, Q^*)$.

When (C2*) is satisfied the partially amalgamated factors will have detachments that satisfy (C2).

We begin with (C1*). Let the *set-discrepancy* of the partially amalgamated factors be defined by

$$\delta_s^* = \sum_{j=1}^{r} \sum_{P^* \in B} |p^*(P^*, V_j) - q^*(P^*, V_j)|.$$

When $\delta_s^* = 0$, (C1*) is satisfied. We must alter the partially amalgamated factors so that δ_s^* is reduced if it is greater than zero.

As we noted, each pair of sets of split vertices in U^* is joined the correct number of times. Thus, for $1 \leq j \leq r$, for each set of split vertices $V_z^* \subseteq B$,

$$\sum_{P^* \in V_z^*} p^*(P^*, V_j^*) = \sum_{P^* \in V_z^*} q^*(P^*, V_j^*). \tag{1}$$

If $\delta_s^* \neq 0$, then there is a vertex $P^* \in B$ and a set of split vertices $V_{z_1}^*$ such that $p^*(P^*, V_{z_1}^*) \neq q^*(P^*, V_{z_1}^*)$. By (1), we can assume without loss of generality that

$$p^*(P^*, V_{z_1}^*) > q^*(P^*, V_{z_1}^*) \tag{2}$$

and that there exists another vertex $Q^* \in B$ that is in the same set of split vertices as P^* such that

$$p^*(Q^*, V^*_{z_1}) < q^*(Q^*, V^*_{z_1}). \tag{3}$$

From $S^*_i(P^*, Q^*)$, $1 \leq i \leq t$, we create a further set $S^*(P^*, Q^*)$: for $1 \leq i \leq t$, if $(u, v) \in S^*_i(P^*, Q^*)$, then $(i, u, v) \in S^*(P^*, Q^*)$. Note that there is an obvious one-to-one relationship between the neighbours, over all the partially amalgamated factors, of P^* and the triples of $S^*(P^*, Q^*)$. Similarly for the neighbours of Q^*.

Claim 6 *There is a sequence of sets of split vertices*

$$\Gamma = V^*_{z_1}, V^*_{z_2}, \ldots, V^*_{z_m}$$

such that

(D1) $V^*_{z_\alpha} \neq V^*_{z_\beta}$ *if* $\alpha \neq \beta$,

(D2) *either* $p^*(P^*, V^*_{z_m}) < q^*(P^*, V^*_{z_m})$ *or* $p^*(Q^*, V^*_{z_m}) > q^*(Q^*, V^*_{z_m})$, *and*

(D3) *for* $2 \leq j \leq m$, *there is a triple* $(i_j, u_j, v_j) \in S^*(P^*, Q^*)$ *where* $u_j \in V^*_{z_{j-1}}$ *and* $v_j \in V^*_{z_j}$.

Proof: In fact, we shall prove that there is a sequence of sets of split vertices

$$\Delta = V^*_{g_1}, V^*_{g_2}, \ldots, V^*_{g_{m'}}$$

such that

(E1) $V^*_{g_1} = V^*_{z_1}$,

(E2) $V^*_{g_\alpha} \neq V^*_{g_\beta}$ if $\alpha \neq \beta$,

(E3) either $p^*(P^*, V^*_{g_{m'}}) < q^*(P^*, V^*_{g_{m'}})$ or $p^*(Q^*, V^*_{g_{m'}}) > q^*(Q^*, V^*_{g_{m'}})$, and

(E4) for $2 \leq j \leq m'$, there is a triple $(i_j, u_j, v_j) \in S^*(P^*, Q^*)$ where $u_j \in V^*_{g_h}$ for some $h \in \{1, 2, \ldots, j-1\}$ and $v_j \in V^*_{g_j}$.

It is easy to see that Δ has a subsequence that has $V^*_{g_1} = V^*_{z_1}$ as the first term and satisfies (D1), (D2) and (D3). (Let $V^*_{g_{m'}}$ be the final term and work backwards. If $V^*_{g_\alpha}$ is the last term reached, then if $\alpha = 1$ the subsequence is found. Otherwise, by (E4), there is a triple $(i_\alpha, u_\alpha, v_\alpha)$. Let the previous term of the sequence be the set of split vertices $V^*_{g_\beta}$ that contains u_α. As $\beta < \alpha$ we must eventually get back to $V^*_{g_1}$.)

We find Δ. The first term $V^*_{g_1} = V^*_{z_1}$ was found before the claim was stated. Suppose that we have found the first μ terms, and that this sequence of μ terms satisfies (E1), (E2) and (E4) with $m' = \mu$. If for any $\alpha \in \{1, 2, \ldots, \mu\}$

$$\begin{aligned} p^*(P^*, V^*_{g_\alpha}) &< q^*(P^*, V^*_{g_\alpha}), \text{ or} \\ p^*(Q^*, V^*_{g_\alpha}) &> q^*(Q^*, V^*_{g_\alpha}), \end{aligned}$$

then we pick the smallest such α and let $\Delta = V^*_{g_1}, V^*_{g_2}, \ldots, V^*_{g_\alpha}$ as this also satisfies (E3). Otherwise, for $1 \le j \le \mu$,

$$p^*(P^*, V^*_{g_j}) \ \ge \ q^*(P^*, V^*_{g_j}), \tag{4}$$

$$p^*(Q^*, V^*_{g_j}) \ \le \ q^*(Q^*, V^*_{g_j}). \tag{5}$$

Let $W = V^*_{g_1} \cup V^*_{g_2} \cup \cdots \cup V^*_{g_\mu}$. As P^* and Q^* are in the same set of split vertices, $q^*(P^*, V^*_j) = q^*(Q^*, V^*_j)$, $1 \le j \le r$. By (2), (3), (4) and (5), over all the factors P^* has more neighbours than Q^* in W. In $S^*(P^*, Q^*)$ there is a triple corresponding to each neighbour of P^* in each factor; similarly there is a triple corresponding to each neighbour of Q^*. So there is a triple $(i_{\mu+1}, u_{\mu+1}, v_{\mu+1}) \in S^*(P^*, Q^*)$, such that $u_{\mu+1} \in W$ and $v_{\mu+1} \notin W$. Let the set of split vertices containing $v_{\mu+1}$ be $V^*_{g_{\mu+1}}$. Then $V^*_{g_{\mu+1}} \ne V^*_{g_j}$, $1 \le j \le \omega$, since $V^*_{g_{\mu+1}} \not\subseteq W$.

We must eventually find a set of split vertices that satisfies (E3): note that

$$\sum_{j=1}^{r} p^*(P^*, V^*_j) = \sum_{j=1}^{r} q^*(P^*, V^*_j), \tag{6}$$

since both sums are equal to $n^2(s-1)$, the sum of the degrees of P^* taken over all the factors. As $p^*(P^*, V^*_{z_1}) > q^*(P^*, V^*_{z_1})$, there is at least one set of split vertices V_z such that $p^*(P^*, V^*_z) < q^*(P^*, V^*_z)$ and therefore V_z, at least, satisfies (E3). This completes the proof of Claim 6. $\square$

We use the claim to reduce δ^*_s. For $2 \le j \le m$, (P^*, Q^*)-swap u_j and v_j

in $F_{i_j}^*$. Each new partially amalgamated factor F_i^* obtained in this way is an l_i-edge-connected detachment of the corresponding colour class H_i.

For $2 \le j \le m-1$, an edge from P^* to a vertex, u_{j+1}, that is in $V_{z_j}^*$, has been deleted and an edge from P^* to a vertex, v_j, that is in $V_{z_j}^*$ has been added. Thus $p^*(P^*, V_{z_j}^*)$ is unchanged. Similarly $p^*(Q^*, V_{z_j}^*)$, $2 \le j \le m-1$, is unchanged.

The edge P^*u_2 is deleted so $p^*(P^*, V_{z_1}^*)$ is reduced by 1. Hence, by (2), δ_s^* is also reduced by 1. The addition of Q^*u_2 causes $p^*(Q^*, V_{z_1}^*)$ to increase by 1 so, by (3), δ_s^* decreases further by 1.

Consider (D2). If $p^*(P^*, V_{z_m}^*) < q^*(P^*, V_{z_m}^*)$, then the addition of P^*v_m causes $p^*(P^*, V_{z_m}^*)$ to increase by 1, and δ_s^* is reduced further by 1. The deletion of Q^*v_m may cause δ_s^* to increase by 1, but at worst δ_s^* is reduced by 2 overall. The other possibility is that $p^*(Q^*, V_{z_m}^*) > q^*(Q^*, V_{z_m}^*)$, and by a similar argument δ_s^* is reduced overall by at least 2 in this case also. Note that the partially amalgamated factors remain loopless.

By repeated application of Claim 6, δ_s^* is reduced to zero. Thus (C1*) is satisfied, that is, every independent set of split vertices in B is joined the correct number of times to every set of split vertices. Independent sets of split vertices in A were already joined the correct number of times to each set of split vertices, so by finding detachments F_i of each F_i^* we could obtain a set of factors that satisfies (C1). For now however, we continue to work with the partially amalgamated factors. We show that when (C1*) is satisfied, we can further alter them so that (C2*) is also satisfied, that is, so that each pair of independent sets of split vertices in B is joined the correct number of times (remember that the independent sets of split vertices in B are just its vertices).

Let the *independent-set-discrepancy* δ_i^* of the partially amalgamated factors be defined by

$$\delta_i^* = \sum_{\substack{P^*, Q^* \in B \\ Q^* \neq P^*}} |p^*(P^*, Q^*) - q^*(P^*, Q^*)|.$$

If (C2*) is satisfied, then $\delta_i^* = 0$. We describe a method that will reduce δ_i^* if it is greater than zero.

We need only consider sets of split vertices that each contain at least two vertices in B since if a vertex $P^* \in B$ is the only vertex in a set of split vertices, then, by (C1*) it is already joined the correct number of times to every other vertex in B.

Claim 7 *Suppose that P^* and Q^* are vertices in B in the same set of split vertices and that $I^*_{z_1} \notin \{P^*, Q^*\}$ is an independent set of split vertices such that*

$$p^*(P^*, I^*_{z_1}) > q^*(P^*, I^*_{z_1}), \tag{7}$$
$$p^*(Q^*, I^*_{z_1}) < q^*(Q^*, I^*_{z_1}). \tag{8}$$

Let $S^*(P^*, Q^*)$ be defined as before. Then there is a sequence of independent sets of split vertices

$$\Gamma = I^*_{z_1}, I^*_{z_2}, \ldots, I^*_{z_m}$$

such that

(F1) $I^*_{z_j} \notin \{P^*, Q^*\}$, $1 \le j \le m$,

(F2) $I^*_{z_\alpha} \neq I^*_{z_\beta}$ *if* $\alpha \neq \beta$,

(F3) *either* $p^*(P^*, I^*_{z_m}) < q^*(P^*, I^*_{z_m})$ *or* $p^*(Q^*, I^*_{z_m}) > q^*(Q^*, I^*_{z_m})$, *and*

(F4) *for* $2 \le j \le m$, *there is a triple* $(i_j, u_j, v_j) \in S^*(P^*, Q^*)$ *where* $u_j \in I^*_{z_{j-1}}$ *and* $v_j \in I^*_{z_j}$.

Proof: Again we shall actually prove that there is a sequence of independent sets of split vertices

$$\Delta = I^*_{g_1}, I^*_{g_2}, \ldots, I^*_{g_{m'}}$$

such that

(G1) $I^*_{g_1} = I^*_{z_1}$,

(G2) $I^*_{g_j} \notin \{P^*, Q^*\}$, $1 \le j \le m$,

(G3) $I^*_{g_\alpha} \neq I^*_{g_\beta}$ if $\alpha \neq \beta$,

(G4) either $p^*(P^*, I^*_{g_{m'}}) < q^*(P^*, I^*_{g_{m'}})$ or $p^*(Q^*, I^*_{g_{m'}}) > q^*(Q^*, I^*_{g_{m'}})$, and

(G5) for $2 \le j \le m'$, there is a triple $(i_j, u_j, v_j) \in S^*(P^*, Q^*)$ where $u_j \in I^*_{g_h}$ for some $h \in \{1, 2, \ldots, j-1\}$ and $v_j \in I^*_{g_j}$.

As before, from Δ we can find Γ.

The first term of Δ, $I^*_{g_1} = I^*_{z_1}$, is known by the hypothesis. Suppose that we have found the first μ terms. If the sequence is not complete, then we can assume that, for $1 \leq j \leq \mu$,

$$\begin{aligned} p^*(P^*, I^*_{g_j}) &\geq q^*(P^*, I^*_{g_j}), && (9) \\ p^*(Q^*, I^*_{g_j}) &\leq q^*(Q^*, I^*_{g_j}). && (10) \end{aligned}$$

Let $W = I^*_{g_1} \cup I^*_{g_2} \cup \cdots \cup I^*_{g_\mu}$. As P^* and Q^* are both vertices in B, $q^*(P^*, I^*) = q^*(Q^*, I^*)$, for every independent set of split vertices $I^* \notin \{P^*, Q^*\}$. Therefore, by (7), (8), (9) and (10), over all the partially amalgamated factors P^* has more neighbours than Q^* in W. So there is a triple $(i_{\mu+1}, u_{\mu+1}, v_{\mu+1}) \in S^*(P^*, Q^*)$ such that $u_{\mu+1} \in W$ and $v_{\mu+1} \notin W$. Let the independent set of split vertices containing $v_{\mu+1}$ be $I^*_{g_{\mu+1}}$. Then $I^*_{g_{\mu+1}} \neq I^*_{g_j}$, $1 \leq j \leq \mu$, since $I^*_{g_{\mu+1}} \not\subset W$, and $I^*_{g_{\mu+1}} \notin \{P^*, Q^*\}$ since $v_{\mu+1} \notin \{P^*, Q^*\}$ as $v_{\mu+1} = P^*$ would imply that $u_{\mu+1} = Q^*$, and $v_{\mu+1} = Q^*$ would imply that there is a loop on Q^*.

We must eventually find a set of split vertices that satisfies (G4): note that

$$\sum p^*(P^*, I^*) = \sum q^*(P^*, I^*), \tag{11}$$

(where the sums are over all independent sets of split vertices I^*) since both sums are equal to $n^2(s-1)$, the sum of the degrees of P^* taken over all the factors. As $p^*(P^*, I^*_{z_1}) > q^*(P^*, I^*_{z_1})$, there is at least one independent set of split vertices I^* such that $p^*(P^*, I^*) < q^*(P^*, I^*)$ and therefore I^*, at least, satisfies (F3). This completes the proof of Claim 7. $\square$

We describe how to use the claim to reduce δ^*_i.

Choose a set of split vertices $V^*_z \subseteq B$ such that

(C1*a) for every independent set of split vertices $I^* \in B \setminus V^*_z$, $p^*(I^*, V^*_j) = q^*(I^*, V^*_j)$, $1 \leq j \leq r$.

As (C1*) implies (C1*a) we can begin by choosing any set as V^*_z. If possible choose a pair of independent sets of split vertices $P^* \in V^*_z$, $I^*_{z_1} \not\subseteq V^*_z$ that satisfies (7). By (C1*a), there exists $Q^* \in V^*_z$ that satisfies (8). Now we can use Claim 7. For $2 \leq j \leq m$, (P^*, Q^*)-swap (u_j, v_j) in $F^*_{i_j}$. Thus for $2 \leq j \leq m-1$, we add $P^* v_j$ to $F^*_{i_j}$ and delete $P^* u_{j+1}$ from $F^*_{i_{j+1}}$, and so $p^*(P^*, I^*_{z_j})$ is unchanged since $v_j, u_{j+1} \in I^*_{z_j}$. Similarly $p^*(Q^*, I^*_{z_j})$ is unchanged, $2 \leq j \leq m-1$. By (7) and (8), the deletion of P^*u_2 and the addition of Q^*u_2 reduce δ_i^* by 2, and, by (F4), the addition of P^*v_m and the deletion of Q^*v_m at worst have no further effect on δ_i^*. Note that no loops are created.

Consider how these (P^*, Q^*)-swaps affect δ_s^*. Let $V_{z_j}^*$ be the set of split vertices that contains $I_{z_j}^*$, $1 \leq j \leq m$. For $2 \leq j \leq m-1$, $p^*(P^*, I_{z_j}^*)$ and $p^*(Q^*, I_{z_j}^*)$ were unchanged so $p^*(P^*, V_{z_j}^*)$ and $p^*(Q^*, V_{z_j}^*)$ do not change. Note that

$$p^*(P^*, V_{z_1}^*) \text{ and } p^*(Q^*, V_{z_m}^*) \text{ are reduced by 1, and} \tag{12}$$

$$p^*(P^*, V_{z_m}^*) \text{ and } p^*(Q^*, V_{z_1}^*) \text{ are increased by 1.} \tag{13}$$

Note that $V_{z_1}^*$ and $V_{z_m}^*$ are both subsets of B since they contain I_{z_1} and I_{z_m} which satisfy (7) and (F4) respectively and we know that each independent set of split vertices in A is already joined the correct number of times to P^* and Q^*.

As $P^*, Q^* \subset V_z^*$, (C1*a) remains satisfied. So we look for further pairs $P^* \in V_z$, $I_{z_1}^* \not\subset V_z$, and repeat the process. When no such pairs remain we have $p^*(P^*, I^*) = q^*(P^*, I^*)$ for every $P^* \in V_z^*$, for every independent set of vertices $I^* \notin V_z^*$. As $p^*(P^*, V_j^*) = \sum p^*(P^*, I^*)$ (where the sum is over all independent sets of vertices $I^* \subseteq V_j^*$), we have $p^*(P^*, V_j^*) = q^*(P^*, V_j^*)$, $1 \leq j \leq r$, $j \neq z$. By (6), this implies that $p^*(P^*, V_z^*) = q^*(Q^*, V_z^*)$ also. Thus

(C1*b) for every vertex $P^* \in V_z^*$, $p^*(P^*, V_j^*) = q^*(P^*, V_j^*)$, $1 \leq j \leq r$.

Note that (C1*a) and (C1*b) together imply (C1*).

Now find a pair $P^* \in V_z^*$, $I_{z_1}^* \in V_z^*$ that satisfies (7). By (C1*b), there exists $Q^* \in V_z^*$ that satisfies (8) so we can reduce δ_i^* further using the claim and the method of (P^*, Q^*)-swapping just described. Note that $V_{z_1}^* = V_z^*$ and that $V_{z_m}^* = V_z^*$ (since only $I_{z_m}^* \in V_z^*$ can satisfy (F4)). Thus (12) and (13) cancel each other out and (C1*a) and (C1*b) remain satisfied. We repeat this until there are no further pairs $P^*, I_{z_1}^* \in V_z^*$ that satisfy (7). Then we begin the whole process again with another choice of V_z^*. Eventually δ_i^* is reduced to zero and (C2*) is satisfied.

Therefore we can find detachments of the partially amalgamated factors that form a set of factors that satisfy (C2), and it is these we work with for the rest of the proof.

Whether or not independent sets of split vertices belong to the same set of split vertices is not important in the next two stages of the proof. Therefore we can label the independent sets of split vertices more simply as $I_1, I_2, \ldots, I_{r'}$.

By (C2), for $1 \le j < z \le r'$,

$$p(I_j, I_z) = q(I_j, I_z). \tag{14}$$

Let the *independent-set-discrepancy* of the factors be defined by

$$\delta_i = \sum_{a \in V(K_n^{(s)})} \sum_{j=1}^{r'} |p(a, I_j) - q(a, I_j)|.$$

When (C3) is satisfied, $\delta_i = 0$. If $\delta_i > 0$, we must show how to reduce it.

Let j and z be fixed. By (14),

$$\sum_{a \in I_z} p(a, I_j) = \sum_{a \in I_z} q(a, I_j). \tag{15}$$

If $\delta_i > 0$, then for some vertex a and some z_1, $p(a, I_{z_1}) \ne q(a, I_{z_1})$. We can assume that

$$\begin{aligned} p(a, I_{z_1}) &> q(a, I_{z_1}), \quad (16)\\ p(b, I_{z_1}) &< q(b, I_{z_1}), \quad (17) \end{aligned}$$

where b is a vertex in the same independent set of split vertices as a.

By Lemma 2, for each F_i we can form an (a, b, l_i)-swap-set which we call $S_i(a, b)$. We form a further set $S(a, b)$: for $1 \le i \le t$, if $(c, d) \in S_i(a, b)$, then $(i, c, d) \in S(a, b)$. Thus $S(a, b)$ contains ordered triples (i, c, d) where c is a neighbour of a and d is a neighbour of b in F_i. Note that there is an obvious one-to-one relationship between the triples of $S(a, b)$ and the neighbours, over all the factors, of a, and also between the triples of $S(a, b)$ and the neighbours, over all the factors, of b.

Claim 8 *There is a sequence of independent sets of split vertices*

$$\Gamma = I_{z_1}, I_{z_2}, \ldots, I_{z_m}$$

such that

(H1) $I_{z_\alpha} \neq I_{z_\beta}$ *if* $\alpha \neq \beta$,

(H2) *either* $p(a, I_{z_m}) < q(a, I_{z_m})$ *or* $p(b, I_{z_m}) > q(b, I_{z_m})$, *and*

(H3) *for* $2 \leq j \leq m$, *there is a triple* $(i_j, c_j, d_j) \in S(a, b)$ *where* $c_j \in I_{z_{j-1}}$ *and* $d_j \in I_{z_j}$.

Proof: In fact we shall prove that there is a sequence of independent sets of split vertices

$$\Delta = I_{g_1}, I_{g_2}, \ldots, I_{g_{m'}}$$

such that

(I1) $I_{g_1} = I_{z_1}$,

(I2) $I_{g_\alpha} \neq I_{g_\beta}$ if $\alpha \neq \beta$,

(I3) either $p(a, I_{g_{m'}}) < q(a, I_{g_{m'}})$ or $p(b, I_{g_{m'}}) > q(b, I_{g_{m'}})$, and

(I4) for $2 \leq j \leq m'$, there is a triple $(i_j, c_j, d_j) \in S(a, b)$ where $c_j \in I_{g_h}$ for some $h \in \{1, 2, \ldots, j-1\}$ and $d_j \in I_{g_j}$.

From Δ we can find Γ.

The first term $I_{g_1} = I_{z_1}$ was found before the claim was stated. If the sequence is not complete, then we can assume that, for $1 \leq j \leq \mu$,

$$p(a, I_{g_j}) \geq q(a, I_{g_j}), \tag{18}$$

$$p(b, I_{g_j}) \leq q(b, I_{g_j}). \tag{19}$$

Let $W = I_{g_1} \cup I_{g_2} \cup \cdots \cup I_{g_\mu}$. As a and b are in the same set of split vertices, $q(a, I_j) = q(b, I_j)$, $1 \leq j \leq r$. Therefore, by (16), (17), (18) and (19) over all the factors a has more neighbours than b in W. So there is a triple $(i_{\mu+1}, c_{\mu+1}, d_{\mu+1}) \in S(a, b)$, such that $c_{\omega+1} \in W$ and $d_{\mu+1} \notin W$. Let the set of split vertices containing $d_{\mu+1}$ be $V_{g_{\mu+1}}$. Then $V_{g_{\mu+1}} \neq V_{g_j}$, $1 \leq j \leq \mu$, since $V_{g_{\mu+1}} \not\subset W$.

We will eventually find a set of split vertices that satisfies (I3): note that

$$\sum_{j=1}^{r} p(a, I_j) = \sum_{j=1}^{r} q(a, I_j), \tag{20}$$

since both sums are equal to $n(s-1)$, the sum of the degrees of a taken over all the factors. As $p(a, V_{z_1}) > q(a, V_{z_1})$, there is at least one set of split vertices V_z such that $p(a, V_z) < q(a, V_z)$ and therefore V_z, at least, satisfies (I3). This completes the proof of Claim 8. $\square$

For $2 \leq j \leq m$, we (a, b)-swap c_j and d_j in F_{i_j}. Each new factor F_i obtained is clearly k_i-regular and, by Lemma 5, it is l_i-edge-connected. It is also a detachment of the corresponding colour class H_i.

For $2 \leq j \leq m-1$, $p(a, I_{z_j})$ and $p(b, I_{z_j})$ are unchanged. By (16) and (17) the reduction in $p(a, I_{z_1})$ and the increase in $p(b, I_{z_1})$ reduce δ_i by 2. By (H2) the changes in $p(a, I_{z_m})$ and $p(b, I_{z_m})$ at worst have no effect on δ_i. The factors remain loopless.

Finally we alter the factors so that (C4) is satisfied

Let the *vertex-discrepancy* of the factors be defined by

$$\delta_v = \sum_{ac \in E(K_n^{(s)})} |p(a, c) - 1|.$$

If (C4) is satisfied, then $\delta_v = 0$. If $\delta_v > 0$, then we show how to reduce it.

We need only consider independent sets of split vertices that each contain at least two vertices: let I_z be an independent set of split vertices that contains just one vertex c. Let a be a vertex in a different part. As (C3) is satisfied, $p(a, I_z) = q(a, I_z) = 1$. As $p(a, c) = p(a, I_z)$, we already have $p(a, c) = 1$.

Claim 9 *Suppose that a and b are vertices in the same independent set of split vertices, that $c_1 \notin \{a, b\}$ and that*

$$\begin{aligned} p(a, c_1) &> 1, & (21) \\ p(b, c_1) &< 1. & (22) \end{aligned}$$

Let $S(a, b)$ be defined as before. Then there is a sequence of vertices $c_1, c_2, \ldots, c_m$ such that

(J1) $c_j \notin \{a, b\}$, $2 \leq j \leq m$,

(J2) $c_\alpha \neq c_\beta$ *if* $\alpha \neq \beta$,

(J3) *either* $p(a, c_m) < 1$ *or* $p(b, c_m) > 1$, *and*

(J4) *for* $1 \leq j \leq m-1$ *there is a triple* $(i_j, c_j, c_{j+1}) \in S(a, b)$.

Proof: The first term of the sequence is known by the hypothesis. If the sequence is not complete, then we can assume, for $1 \leq j \leq \mu$,

$$\begin{aligned} p(a, c_j) &\geq 1, \\ p(b, c_j) &\leq 1. \end{aligned}$$

As $p(a, c_\mu) \geq 1$ we can find a triple $(i_\mu, c_\mu, c_{\mu+1}) \in S(a, b)$. As there are no loops and $c_{\mu+1}$ is a neighbour of b, $c_{\mu+1} \neq b$. By (J1), $c_\mu \neq b$ and a is the second element of a pair in $S_{i_\mu}(a, b)$ only if b is the first element, so $c_{\mu+1} \neq a$. By (22), $p(b, c_1) = 0$, so $c_{\mu+1} \neq c_1$. As $p(b, c_j) \leq 1$, $2 \leq j \leq \mu$, there is at most one triple in $S(a, b)$ with c_j as the third element and we have already found one such triple (namely (i_{j-1}, c_{j-1}, c_j)). Therefore $c_{\mu+1} \neq c_j$, $2 \leq j \leq \mu$.

The sequence must terminate: there is a finite number of vertices and it is easily seen that $p(a, c_1) > 1$ implies that for some vertex c, $p(a, c) < 1$. This completes the proof of Claim 9. □

We describe how to use the claim to reduce the vertex-discrepancy. First choose an independent set of split vertices I_z such that

(C3a) for every vertex $c \notin I_z$, $p(c, I_j) = q(c, I_j)$, $1 \leq j \leq r$.

As (C3) implies (C3a) we can initially choose any set of split vertices as I_z. If possible choose a pair of vertices $a \in I_z$, $c_1 \notin I_z$ that satisfy (21). By (C3a) there is a vertex $b \in I_z$ that satisfies (22). Therefore we use Claim 9: for $1 \leq j \leq m-1$, (a, b)-swap (c_j, c_{j+1}) in F_{i_j}. For $2 \leq j \leq m-1$, $p(a, c_j)$ and $p(b, c_j)$ are unchanged. By (21), the deletion of ac_1 reduces δ_v by 1, and, by (22), the addition of bc_1 reduces δ_v further by 1. By (J3), the addition of ac_m and the deletion of bc_m at worst has no net effect on δ_v. So overall δ_v is reduced by at least 2. As $c_j \notin \{a, b\}$, $1 \leq j \leq m$, no loops are created.

Consider the effect of these (a, b)-swaps on δ_i. Let I_{z_j} be the set of split vertices that contains c_j, $1 \leq j \leq m$. For $2 \leq j \leq m-1$, $p(a, c_j)$ and $p(b, c_j)$ were unchanged so $p(a, I_{z_j})$ and $p(b, I_{z_j})$ are unchanged. Note that

$$p(a, I_{z_1}) \text{ and } p(b, I_{z_m}) \text{ are reduced by 1, and} \tag{23}$$

$$p(a, I_{z_m}) \text{ and } p(b, I_{z_1}) \text{ are increased by 1.} \tag{24}$$

As $a, b \in I_z$, (C1a) remains satisfied (even though (C1) does not). So we can look for further pairs $a \in I_z$, $c_1 \notin V_z$ that satisfy (21) and repeat the process.

When no such pairs remain we have $p(a,c)=1$ for every $a \in I_z$, $c \notin I_z$. For $1 \leq j \leq r$, $j \neq z$, $p(a,I_j) = \sum_{c \in I_j} p(a,c) = |I_j|$. Thus $p(a,I_j) = q(a,I_j)$, $1 \leq j \leq r$, $j \neq z$. By (20), this implies that $p(a,I_z) = q(a,I_z)$ also. Thus

(C3b) for every vertex $a \in I_z$, $p(a,I_j) = q(a,I_j)$, $1 \leq j \leq r$.

Note that (C3a) and (C3b) together imply (C3).

Now if possible choose a pair $a \in I_z$, $c \in I_z$ that satisfies (21). By (C3b), there is a vertex $b \in I_z$ that satisfies (22), so we can use the claim to reduce δ_v further. Note that $I_{z_1} = I_z$ (since I_{z_1} is the set that contains c_1). Note also that that $I_{z_m} = I_z$ since $c_m \in I_{z_m}$ and I_m satisfies (J3) and we know that $p(a,c)=1$ for all $a \in I_z$, $c \notin V_z$. Thus (23) and (24) cancel each other out and (C3a) and (C3b) remain satisfied. Look for further pairs $a, c_1 \in I_z$ that satisfy (21) and reduce δ_v further. When no such pairs remain (C3) is satisfied since (C3a) and (C3b) are satisfied, and we can begin the process again with another choice of I_z. Eventually δ_v is reduced to zero and (C4) is satisfied. This completes the proof of Theorem 4. $\square$

2.6 Proof of Theorem 1

The following four sentences prove the necessity of the four conditions. The degree of a vertex in $K_n^{(s)}$ is equal to the sum of its degrees in the factors. By the handshaking lemma, a regular graph on an odd number of vertices must have even degree. The set of all edges incident with a vertex form an edge-cutset. A 1-factor of a simple graph (other than K_2) is not connected.

Now we have to show that there exists a (t,K,L)-factorization of $K_n^{(s)}$ whenever (A1) to (A4) are satisfied. By Theorem 4, unless $l_i = 1$ for some i it is sufficient to find an outline (t,K,L)-factorization of $K_n^{(s)}$ that satisfies (Z1) and (Z2). It easy to find such outline factorizations H, f and f_h, $1 \leq h \leq s$. Let $V(H) = \{v\}$. Let there be $n^2\binom{s}{2}$ loops on v (this is the number of edges in $K_n^{(s)}$). Let $nsk_i/2$ of the loops be coloured i, $1 \leq i \leq t$. Let $f(v) = ns$ and let $f_h(v) = n$, $1 \leq h \leq s$. It is easy to see that H, f and f_h, $1 \leq h \leq s$, satisfy (B1) to (B5).

Now for the case where some $l_i = 1$. Replace every instance of 1 in L with 2 to obtain L'. Note that, by (A4), if $l_i = 1$, then $k_i \geq 2$ so t, K and L' satisfy (A1) to (A4). A (t,K,L')-factorization is also a (t,K,L)-factorization since l_i prescribes only the *minimum* edge-connectivity. $\square$

3 Embedding factorizations

The most general embedding result that we might aim to find would show when it is possible to find an embedding of a factorization of $G = K_{a_1,\dots,a_s}$ in a (t, K, L)-factorization of $K_n^{(s)}$, where $a_i \le n$, $1 \le i \le s$. To prove this using amalgamations however, we would have to add one vertex v_0 to G to create an outline (t, K, L)-factorization of $K_n^{(s)}$. Thus we would have $f_h(v_0) = n - a_h$, $1 \le h \le s$. But if we are to use Theorem 4 we require, by (Z1) and (Z2), that $f_h(v_0) \in \{0, n\}$, $1 \le h \le s$. In Theorem 11, we find a way around this difficulty in the bipartite case and show when a factorization of $K_{a,b}$ can be embedded in a (t, K, L)-factorization of $K_{n,n}$, $a, b \le n$. In the general case however, we confine ourselves to the following: in Theorem 10 we show precisely when a factorization $G_1, \dots, G_t$ of $K_n^{(\sigma)}$ can be embedded in a (t, K, L)-factorization $F_1, \dots, F_t$ of $K_n^{(s)}$ (except that we again have the restriction $l_i \ne 1$, $1 \le i \le t$). This has only been proved previously for the case of Hamiltonian decompositions [4].

3.1 Embedding equipartite graphs

We need some definitions before we can state the theorem. Let ω_i be the number of connected components of G_i and let these components be $C_{i,1}, \dots, C_{i,\omega_i}$. Let $\varepsilon_{i,j} = \sum_{v \in V(C_{i,j})} k_i - d_{G_i}(v)$, and let $\varepsilon_i = \sum_{j=1}^{\omega_i} \varepsilon_{i,j}$. Let $r_{i,j}$ be the number of minimal separating sets of $C_{i,j}$ that contain fewer than l_i edges, let these sets be $E_1^{i,j}, E_2^{i,j}, \dots, E_{r_{i,j}}^{i,j}$, and let $C_{m_1}^{i,j}$ and $C_{m_2}^{i,j}$ be the connected components of $C_{i,j} - E_m^{i,j}$. Let $\varepsilon_{i,j,m_p} = \sum_{v \in V(C_{m_p}^{i,j})} k_i - d_{G_i}(v)$.

Theorem 10 *Suppose that n, s, t, K and L are such that a (t, K, L)-factorization of $K_n^{(s)}$ exists and that $l_i \ne 1$, $1 \le i \le t$. Let $\alpha = n(s - \sigma)$. A t-edge-coloured $K_n^{(\sigma)}$ can be embedded in a (t, K, L)-factorization of $K_n^{(s)}$ if and only if*

(I) $d_{G_i}(v) \le k_i$ *for each* $v \in V(K_n^{(\sigma)})$, *for* $1 \le i \le t$,

(II) $\varepsilon_{i,j} \ge l_i$ *for* $1 \le i \le t$, $1 \le j \le \omega_i$,

(III) $\alpha \ge \max\{\varepsilon_i / k_i : 1 \le i \le t\}$, *and*

(IV) $\varepsilon_{i,j,m_p} \geq l_i - |E^{i,j}_m|$, *for* $1 \leq i \leq t$, $1 \leq j \leq \omega_i$, $1 \leq m \leq r_{i,j}$, $1 \leq p \leq 2$.

Proof: By Theorem 1, we may assume that conditions (A1) to (A4) are satisfied.

Necessity: suppose that a t-edge-coloured $K_n^{(\sigma)}$ is embedded in an (t, K, L)-factorization of $K_n^{(s)}$. We show that the conditions of the theorem hold.

As G_i is a subgraph of a k_i-regular graph, $d_{G_i}(v) \leq k_i$ for each $v \in V(K_n^{(\sigma)})$, for $1 \leq i \leq t$. So (I) holds.

By definition $\varepsilon_{i,j}$ is the number of edges incident with the vertices of $C_{i,j}$ in $E(F_i) \setminus E(G_i)$. All these edges join $C_{i,j}$ to $V(K_n^{(s)}) \setminus V(K_n^{(\sigma)})$ and form an edge-cutset so there must be at least l_i of them. So (II) holds.

Similarly, ε_i is the number of edges incident with the vertices of G_i in $E(F_i) \setminus E(G_i)$, and all these edges join G_i to one of the α vertices of $V(K_n^{(s)}) \setminus V(K_n^{(\sigma)})$ which each have degree k_i. Thus $\varepsilon_i \leq k_i\alpha$. So (III) holds.

For $1 \leq i \leq t$, $1 \leq j \leq \omega_i$, $1 \leq m \leq r_{i,j}$, there must be l_i edge-disjoint paths from $C^{i,j}_{m_1}$ to $C^{i,j}_{m_2}$. We know that $|E^{i,j}_m|$ of these paths are in $C_{i,j}$. The remainder must go through $V(K_n^{(s)}) \setminus V(K_n^{(\sigma)})$. Therefore there must be at least $l_i - |E^{i,j}_m|$ edges from each of $C^{i,j}_{m_1}$ and $C^{i,j}_{m_2}$ to $V(K_n^{(s)}) \setminus V(K_n^{(\sigma)})$. So (IV) holds as ε_{i,j,m_p} is the number of edges incident with the vertices of $C^{i,j}_{m_p}$ in $E(F_i) \setminus E(G_i)$.

Sufficiency: to complete the proof we must show that if the four conditions hold then we can find an embedding. From $K_n^{(\sigma)}$ we form H, f and f_h, $1 \leq h \leq s$, an outline (t, K, L)-factorization of $K_n^{(s)}$. Let $V(H) = V(K_n^{(\sigma)}) \cup \{v_0\}$. Let $f(v_0) = \alpha$, let $f(v) = 1$ for each $v \in V(K_n^{(\sigma)})$. Let $f_h(v_0) = 0$, $1 \leq h \leq \sigma$, and let $f_h(v_0) = n$, $\sigma + 1 \leq h \leq s$. If $v \in K_n^{(\sigma)}$, then let $f_h(v) = 1$ if $v \in P_h$, else let $f_h(v) = 0$. The edge set of H contains the edges of $K_n^{(\sigma)}$ (which are already coloured) and

- for $1 \leq i \leq t$, for each $v \in K_n^{(\sigma)}$, there are $k_i - d_{G_i}(v)$ edges coloured i from v_0 to v, and
- for $1 \leq i \leq t$, there are $(\alpha k_i - \varepsilon_i)/2$ loops coloured i on v_0.

If we can prove that H, f and f_h, $1 \leq h \leq s$, are an outline (t, K, L)-factorization of $K_n^{(s)}$, then we can apply Theorem 4. Any (t, K, L)-factorization $F_1, \ldots, F_t$ of $K_n^{(s)}$ of which H, f and f_h, $1 \leq h \leq s$, is an amalgamation is such that G_i is a subgraph of F_i.

We check that the number of loops added of each colour is an integer. As $\alpha = n(s-\sigma)$,

$$\begin{aligned}\frac{\alpha k_i - \varepsilon_i}{2} &= \frac{n(s-\sigma)k_i - \varepsilon_i}{2} \\ &= \frac{k_i n s}{2} - \varepsilon_i - \frac{k_i n\sigma - \varepsilon_i}{2}\end{aligned}$$

which is an integer since, by (A2), $k_i ns$ is even and $(k_i n\sigma - \varepsilon_i)/2 = |E(G_i)|$.

We must show that H, f and f_h, $1 \le h \le s$, satisfy (B1) to (B5).

For $v, w \in V(K_n^{(\sigma)})$, there is one edge joining v to w unless they are in the same part. For $v \in V(K_n^{(\sigma)})$, the number of edges from v to v_0 is

$$\begin{aligned}\sum_{i=1}^{t}(k_i - d_{G_i}(v)) &= \sum_{i=1}^{t} k_i - \sum_{i=1}^{t} d_{G_i}(v) \\ &= n(s-1) - n(\sigma-1) \\ &= \alpha \\ &= \sum_{\substack{h_1,h_2\in\{1,\dots,s\}\\ h_1\neq h_2}} f_{h_1}(v) f_{h_2}(v_0).\end{aligned}$$

So (B1) is satisfied.

For $v \in V(K_n^{(\sigma)})$ there are no loops on v. The number of loops on v_0 is

$$\begin{aligned}
\sum_{i=1}^{t} \frac{\alpha k_i - \varepsilon_i}{2} &= \sum_{i=1}^{t} \frac{\alpha k_i}{2} - \sum_{i=1}^{t} \sum_{v \in V(K_n^{(\sigma)})} \frac{k_i - d_{G_i}(v)}{2} \\
&= \frac{\alpha n(s-1)}{2} - \sum_{v \in V(K_n^{(\sigma)})} \sum_{i=1}^{t} \frac{k_i - d_{G_i}(v)}{2} \\
&= \frac{\alpha n(s-1)}{2} - \sum_{v \in V(K_n^{(\sigma)})} \frac{n(s-1) - n(\sigma-1)}{2} \\
&= \frac{\alpha n(s-1)}{2} - \sum_{v \in V(K_n^{(\sigma)})} \frac{\alpha}{2} \\
&= \frac{\alpha n(s-1)}{2} - \frac{\alpha n \sigma}{2} \\
&= \frac{\alpha n(s-1-\sigma)}{2} \\
&= \frac{n^2(s-\sigma)(s-\sigma-1)}{2} \\
&= n^2 \binom{s-\sigma}{2} \\
&= \sum_{1 \le h_1 < h_2 \le s} f_{h_1}(v_0) f_{h_2}(v_0)
\end{aligned}$$

So (B2) is satisfied.

For $v \in V(K_n^{(\sigma)})$ there are $d_{G_i}(v) + (k_i - d_{G_i}(v)) = k_i = k_i f(v)$ edges of each colour incident with v. The number of edges of each colour incident with v_0 is

$$\begin{aligned}
\sum_{v \in V(K_n^{(\sigma)})} (k_i - d_{G_i}(v)) + \alpha k_i - \varepsilon_i &= \varepsilon_i + \alpha k_i - \varepsilon_i \\
&= \alpha k_i \\
&= k_i f(v_0).
\end{aligned}$$

So (B3) is satisfied.

It is easy to see that (B4) is satisfied.

To show that (B5) is satisfied we must show that each H_i has an l_i-edge-connected k_i-regular detachment. Thus we show that each H_i satisfies the conditions of Proposition 2.

First we show that each H_i is l_i-edge-connected. Suppose that H_i is not l_i-edge-connected. Then there is a minimal edge-cutset E such that $|E| < l_i$. As E is minimal it will contain only edges from one component of G_i, say $C_{i,1}$, and perhaps also edges from v_0 to $C_{i,1}$. It cannot contain only edges from v_0 to $C_{i,1}$ since there are $\sum\limits_{v \in V(C_{i,j})} (k_i - d_{G_i}(v)) = \varepsilon_{i,j}$ such edges and, by (II), $\varepsilon_{i,j} \geq l_i$. The edges of E contained in $C_{i,1}$ form one of its minimal separating sets, say $E_1^{i,1}$, and we can assume that the two components of $H_i - E$ are $C_{1_1}^{i,1}$ and $H_i - C_{1_1}^{i,1}$. Therefore E must also contain all the edges from $C_{1_1}^{i,1}$ to v_0. There are $\sum\limits_{v \in V(C_{i,1}^{1_1})} (k_i - d_{G_1}(v)) = \varepsilon_{i,1,1_1}$ such edges. So

$$\begin{aligned} |E| &= |E_1^{i,1}| + \varepsilon_{i,1,1_1} \\ &\geq l_i, \end{aligned}$$

by (IV), a contradiction. So each H_i satisfies (X1).

As $l_i \neq 1$, $1 \leq i \leq t$ we need not consider (X2). Since $n \neq 3$ by assumption, $\alpha \neq 2$ and we need not consider (X3).

Finally, (X4) is satisfied since each H_i contains more than two vertices. □

3.2 Embedding bipartite graphs

We consider an embedding of an edge-coloured $K_{a,b}$ with colour classes $G_1, \ldots, G_t$ in a (t, K, L)-factorization $F_1, \ldots, F_t$ of $K_{n,n}$. As well as the definitions used in Theorem 10 we need the following. For each component $C_{i,j}$ of H_i let $\gamma_{i,j}$ be

$$\min_{\substack{m,x,y \\ x \neq y}} \left\{ |E_m^{i,j}| + \sum_{v \in V(C_{m_1}^{i,j}) \cap P_x} (k_i - d_{G_i}(v)) + \sum_{v \in V(C_{m_2}^{i,j}) \cap P_y} (k_i - d_{G_i}(v)), \sum_{v \in V(C_{i,j}) \cap P_x} (k_i - d_{G_i}(v)) \right\}$$

Note that the two parts of $K_{n,n}$ are P_1 and P_2 where the set of a independent vertices of $K_{a,b}$ are embedded in P_1 and the set of b independent vertices are embedded in P_2.

Theorem 11 *Suppose that $n, s = 2, t, K$ and L are such that a (t, K, L)-factorization of $K_{n,n}$ exists, and that $l_i \neq 1$, $1 \leq i \leq t$. Let a and b be integers, $1 \leq a, b \leq n$. A t-edge-coloured $K_{a,b}$ can be embedded in a (t, K, L)-factorization of $K_{n,n}$ if and only if*

(I) $d_{G_i}(v) \leq k_i$ *for each* $v \in V(K_{a,b})$, *for* $1 \leq i \leq t$,

(II) $\varepsilon_{i,j} \geq l_i$ *for* $1 \leq i \leq t$, $1 \leq j \leq \omega_i$,

(III) $2n - (a + b) \geq \max\{\varepsilon_i/k_i : 1 \leq i \leq t\}$,

(IV) *if* $a = n - 2$ *and* $k_i = l_i$ *is odd, then, for* $1 \leq j \leq \omega_i$, *if there exists* $v \in P_2 \cap C_{i,j}$ *such that* $d_{G_i}(v) < k_i$, *then either there exists* $w \in P_1 \cap C_{i,j}$ *such that* $d_{G_i}(w) < k_i$ *or for all* $u \in P_2 \setminus C_{i,j}$, $d_{G_i}(u) = k_i$.

(V) *if* $b = n - 2$ *and* $k_i = l_i$ *is odd, then, for* $1 \leq j \leq \omega_i$, *if there exists* $v \in P_1 \cap C_{i,j}$ *such that* $d_{G_i}(v) < k_i$, *then either there exists* $w \in P_2 \cap C_{i,j}$ *such that* $d_{G_i}(w) < k_i$ *or for all* $u \in P_1 \setminus C_{i,j}$, $d_{G_i}(u) = k_i$.

(VI) $\sum_{j=1}^{\omega_i} \gamma_{i,j} + [(2n - (a + b))k_i - \varepsilon_i]/2 \geq l_i$, $1 \leq i \leq t$, *and*

(VII) $\varepsilon_{i,j,m_p} \geq l_i - |E^{i,j}_m|$, *for* $1 \leq i \leq t$, $1 \leq j \leq \omega_i$, $1 \leq m \leq r_{i,j}$, $1 \leq p \leq 2$.

Proof: By Theorem 1, we may assume that conditions (A1) to (A4) are satisfied.

Necessity: suppose that a t-edge-coloured $K_n^{(\sigma)}$ is embedded in an (t, K, L)-factorization of $K_n^{(s)}$. We show that the conditions (III), (IV), (V) and (VI) of the theorem hold. The others are identical to conditions of Theorem 10 and the reasons for their are necessity are the same.

Note that ε_i is the number of edges incident with the vertices of G_i in $E(F_i) \setminus E(G_i)$. These edges are all incident with the $2n - (a + b)$ vertices of $V(K_{n,n}) \setminus V(K_{a,b})$ which each have degree k_i. Thus $\varepsilon_i \leq (2n - (a + b))k_i$. So (III) holds.

Suppose that $a = n - 2$, $k_i = l_i$ is odd and there exists $v \in P_2 \cap C_{i,j}$ (for some j) such that $d_{G_i}(v) < k_i$. Thus v is joined to at least one of the two vertices of $P_1 \setminus V(K_{a,b})$. If there is a vertex $u \in P_2 \cap (K_{a,b} \setminus C_{i,j})$ such that $d_{G_i}(u) \neq k_i$ and there is no vertex $w \in P_1 \cap C_{i,j}$ such that $d_{G_i}(w) < k_i$, then the two vertices of $P_1 \setminus K_{a,b}$ form a cutset. Let J_1 and J_2 be the two components obtained when the cutset is removed. There must be k_i paths from J_1 to J_2 through the cutset so each of the two vertices must be joined to each of J_1 and J_2 by $k_i/2$ edges. This is a contradiction since k_i is odd. So (IV) holds. A similar argument shows that (V) holds.

From the argument that shows that (III) holds we can see that in F_i there are $[(2n-(a+b))k_i - \varepsilon_i]/2$ edges joining vertices of $P_1 \setminus V(K_{a,b}) = W_1$ to vertices of $P_2 \setminus V(K_{a,b}) = W_2$. We will form an edge-cutset of F_i that separates W_1 from W_2. First we take all the edges joining vertices of W_1 to vertices of W_2. Next we ensure there is no path from W_1 to W_2 through $C_{i,j}$, $1 \le j \le \omega_i$. We do this by, for each j, taking either all the edges from $C_{i,j}$ to W_1 (or W_2) or taking an edge-cutset $E_m^{i,j}$ from $C_{i,j}$ and also all edges from $C_{m_1}^{i,j}$ to W_1 and from $C_{m_2}^{i,j}$ to W_2 (or vice versa). Thus the minimum number of edges incident with $C_{i,j}$ we must take is $\gamma_{i,j}$, and so the edge-cutset formed has at least $\sum_j \gamma_{i,j} + [(2n-(a+b))k_i - \varepsilon_i]/2$ edges. So (VI) holds.

Sufficiency: to complete the proof we must show that if the conditions hold then we can find an embedding. From $K_{a,b}$ we form H, f and f_h, $1 \le h \le 2$, an outline (t, K, L)-factorization of $K_{n,n}$. Let $V(H) = V(K_{a,b}) \cup \{v_1, v_2\}$. Let $f(v_1) = f_1(v_1) = n-a$; let $f(v_2) = f_2(v_2) = n-b$. For each $v \in K_{a,b}$, let $f_h(v) = 1$, if $v \in P_h$, else let $f_h(v) = 0$. Henceforth when we refer to P_1 and P_2 we will mean vertices in $K_{a,b}$; we do not consider the vertices v_1 and v_2 to be in these parts. The edge set of H contains the edges of $K_{a,b}$ (which are already coloured) and

- for each $v \in P_1$, there are $k_i - d_{G_i}(v)$ edges coloured i from v to v_2,
- for each $v \in P_2$, there are $k_i - d_{G_i}(v)$ edges coloured i from v to v_1, and
- for $1 \le i \le t$, there are $[(2n-(a+b))k_i - \varepsilon_i]/2$ edges coloured i from v_1 to v_2

By (III), the number of edges of each colour from v_1 to v_2 is not negative.

If we can prove that H, f and f_h, $1 \le h \le 2$, are an outline (t, K, L)-factorization of $K_{n,n}$, then we can apply Theorem 4. Any (t, K, L)-factorization $F_1, F_2, \ldots, F_t$ of $K_{n,n}$ of which H, f and f_h, $1 \le h \le s$, is an amalgamation is such that G_i is a subgraph of F_i.

We note that it is a simple matter to form H from the edge-coloured $K_{a,b}$. We add edges so that the vertices of $V(K_{a,b})$ are incident with the correct number of edges of each colour and then add edges between v_1 and v_2 so that the total number of edges of each colour is correct. Most importantly, we do not have to make any choices about how to colour edges.

As an aside, we note this would not be the case if we tried to use the same technique to embed $K_{a_1,\ldots,a_s}$ in $K_n^{(s)}$, $s > 2$. Suppose we add s vertices

$v_1, \dots, v_s$ to form an outline graph where v_i is the vertex that will be split to complete P_i. Now suppose that a vertex $v \in P_1$ in $K_{a_1,\dots,a_s}$ is incident with less than k_1 edges of colour 1. Then in the outline graph, we have to have an edge coloured 1 from v to v_i, $i \neq 1$. So we have a choice of v_i (whereas in the bipartite case we have to choose v_2). So rather than proving that a particular outline graph satisfies (B1) to (B5), we have to show that at least one graph (of all the possible ones we could choose to create) satisfies the conditions.

Back to the proof: we must show that H, f and f_h, $1 \leq h \leq s$, satisfy (B1) to (B5).

For $v, w \in V(K_{a,b})$, there is one edge joining v to w unless they are in the same part. There are no edges from vertices in P_1 to v_1 and from vertices in P_2 to v_2. For $v \in P_1$, the number of edges from v to v_2 is

$$\begin{aligned}\sum_{i=1}^{t}(k_i - d_{G_i}(v)) &= \sum_{i=1}^{t} k_i - \sum_{i=1}^{t} d_{G_i}(v)\\ &= n - b\\ &= \sum_{\substack{h_1,h_2\in\{1,2\}\\ h_1\neq h_2}} f_{h_1}(v) f_{h_2}(v_2).\end{aligned}$$

A similar argument shows that each $v \in P_2$ is joined to v_1 by the correct number of edges. The number of edges from v_1 to v_2 is

$$\begin{aligned}\sum_{i=1}^{t}\frac{(2n-(a+b))k_i - \varepsilon_i}{2} &= \frac{2n-(a+b)}{2}\sum_{i=1}^{t} k_i - \sum_{i=1}^{t}\sum_{v\in V(K_{a,b})}\frac{k_i - d_{G_i}(v)}{2}\\ &= n^2 - \frac{(a+b)n}{2} - \sum_{v\in V(K_{a,b})}\sum_{i=1}^{t}\frac{k_i - d_{G_i}(v)}{2}\\ &= n^2 - \frac{(a+b)n}{2} - \sum_{v\in P_1}\frac{n-b}{2} - \sum_{v\in P_2}\frac{n-a}{2}\\ &= n^2 - \frac{(a+b)n}{2} - \frac{a(n-b)}{2} - \frac{b(n-a)}{2}\\ &= (n-b)(n-a)\\ &= \sum_{\substack{h_1,h_2\in\{1,2\}\\ h_1\neq h_2}} f_{h_1}(v_1) f_{h_2}(v_2).\end{aligned}$$

So (B1) is satisfied.

There are no loops in H so (B2) is satisfied.

For $v \in V(K_{a,b})$, there are $d_{G_i}(v) + (k_i - d_{G_i}(v)) = k_i = k_i f(v)$ edges of each colour incident with v. We must show that v_1 and v_2 are incident with the correct number of edges of each colour. First note that

$$\begin{aligned}\varepsilon_i &= \sum_{v \in V(K_{a,b})} k_i - d_{G_i}(v) \\ &= (a+b)k_i - \sum_{v \in P_1} d_{G_i}(v) - \sum_{v \in P_2} d_{G_i}(v).\end{aligned}$$

Clearly the two sums are equal so

$$\sum_{v \in P_2} d_{G_i}(v) = \frac{(a+b)k_i - \varepsilon_i}{2}.$$

The number of edges coloured i incident with v_1 is

$$\begin{aligned}&\sum_{v \in P_2} (k_i - d_{G_i}(v)) + \frac{(2n - (a+b))k_i - \varepsilon_i)}{2} \\ &= bk_i - \sum_{v \in P_2} d_{G_i}(v) + \frac{(2n - (a+b))k_i - \varepsilon_i)}{2} \\ &= bk_i - \frac{(a+b)k_i}{2} + \frac{\varepsilon_i}{2} + \frac{(2n - (a+b))k_i - \varepsilon_i)}{2} \\ &= (n-a)k_i \\ &= k_i f(v_1)\end{aligned}$$

A similar argument shows that v_2 is incident with $k_i f(v_2)$ edges of colour i. So (B3) is satisfied.

It is easy to see that (B4) is satisfied.

Finally to show that (B5) is satisfied we must show that each H_i has an l_i-edge-connected k_i-regular detachment. Thus we show that each H_i satisfies the conditions of Proposition 2.

First we show that each H_i is l_i-edge-connected. Suppose that H_i is not l_i-edge-connected. Then there is a minimal edge-cutset E such that $|E| < l_i$. We consider two cases. First assume that v_1 and v_2 are in the same component of $H_i - E$. As E is minimal it will contain only edges from one component of G_i, say $C_{i,1}$, and perhaps also edges from v_1 and v_2 to $C_{i,1}$. It cannot contain only edges from v_1 and v_2 to $C_{i,1}$ since there are $\sum_{v\in V(C_{i,j})} (k_i - d_{G_i}(v)) = \varepsilon_{i,j}$ such edges and by (II), $\varepsilon_{i,j} \geq l_i$. The edges of E contained in $C_{i,1}$ form one of its minimal separating sets, say $E_1^{i,1}$, and we can assume that the two components of $H_i - E$ are $C_{1_1}^{i,1}$ and $H_i - C_{1_1}^{i,1}$. Therefore E must also contain all the edges from $C_{1_1}^{i,1}$ to v_1 and v_2. There are $\sum_{v\in V(C_{i,1}^{1_1})} (k_i - d_{G_1}(v)) = \varepsilon_{i,1,1_1}$ such edges. So

$$\begin{aligned} |E| &= |E_1^{i,1}| + \varepsilon_{i,1,1_1} \\ &\geq l_i, \end{aligned}$$

by (VII), a contradiction.

Now assume that v_1 and v_2 are in different components of $H_i - E$. Thus E must contain the $[(2n-(a+b))k_i - \varepsilon_i]/2$ edges from v_1 to v_2. For each component $C_{i,j}$, E contains either all the edges from $C_{i,j}$ to one of v_1 or v_2, or an edge-cutset of $C_{i,j}$, say $E_1^{i,j}$, and all the edges from $C_{1,1}^{i,j}$ to v_1 and from $C_{1,2}^{i,j}$ to v_2 (or vice versa). It follows from (VI) that $|E| \geq l_i$. So each H_i satisfies (X1).

As $l_i \neq 1$, $1 \leq i \leq t$ we need not consider (X2),

H_i has a vertex of degree $2l_i$ only if $a = n-2$ or $b = n-2$. We can see that, by (IV) and (V), these vertices will not be cutvertices so (X3) is satisfied.

Finally, (X4) is satisfied since each H_i contains more than two vertices. $\square$

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