Some results on the Oberwolfach problem

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Abstract

The well-known Oberwolfach problem is to show that it is possible to 2-factorize K_n (n odd) or K_n less a 1-factor (n even) into predetermined 2-factors, all isomorphic to each other; a few exceptional cases where it is not possible are known. In this paper we introduce a completely new technique which enables us to show that there is a solution when each 2-factor consists of k r-cycles and one $(n-kr)$-cycle for all $n \geq 6kr-1$. Solutions are also given (with three exceptions) for all possible values of n when there is one r-cycle, $3 \leq r \leq 9$, and one $(n-r)$-cycle, or when there are are two r-cycles, $3 \leq r \leq 4$, and one $(n-2r)$-cycle.

1 Introduction

Let K_n^* be the complete graph K_n if n is odd and K_n less a 1-factor if n is even. The problem of determining whether there is a 2-factorization of K_n^* in which each 2-factor is isomorphic to the same specified graph is known as the Oberwolfach problem. The notation $\mathrm{OP}(r_1^{a_1}, r_2^{a_2}, \ldots, r_s^{a_s})$ represents the case in which each 2-factor must consist of a_i r_i-cycles, $1 \leq i \leq s$. The problem was formulated by Ringel and is first mentioned in [6]. Many cases have now been solved; see, for example, [1, 2, 7, 11, 12, 13, 14], or, for a summary of known results, [3]. In this paper we obtain some further solutions by a method that is completely novel in the context of the Oberwolfach problem. The method is an adaptation of the outline/amalgamtion technique used, in particular, in [8] concerning Hamiltonian decompositions of complete graphs. Our main result is:

Theorem 1 *Let $r \geq 3$, $k \geq 1$ and $n \geq 6kr-1$ be integers. Then $OP(r^k, n-kr)$ has a solution.*

Theorem 1 is just a specialization of the following more detailed result.

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Theorem 2 *Let $r \geq 3$, $k \geq 1$ and n be integers. Then a solution to OP$(r^k, n - kr)$ exists if*

1. *for even r, either $n \geq 6kr - 3$, or* $n \in \{2r(2k+i) - 3, 2r(2k+i) - 2 \mid i = 1, 2, \ldots, k-1\}$,
2. *for odd r, even k, either $n \geq 6kr - 3$, or $n \in A \cup B$, where* $A = \{2r(2k+2i+1)-1, 2r(2k+2i+1), 2r(2k+2i+2)-3, 2r(2k+2i+2)-2 \mid i = 0, 1, \ldots, \lfloor (k-3)/2 \rfloor\}$, $B = \{2r(3k-1)-1, 2r(3k-1)\}$,
3. *for odd r, odd k, either $n \geq 6kr - 1$, or $n \in A$ (where A is as in part 2),*
4. $n = 4kr - 2$.

Theorem 2 is a consequence of Lemmas 3 and 4, except in the cases $(r, k) = (3, 1)$, $n = 27$ or $n = 28$. These cases are covered by Theorem 5 below.

Lemma 3 *Let $r \geq 3$, $k \geq 1$, $m \geq 2k+1$ and n be integers with $(r, m) \neq (3, 4)$. Then K_n^* has a 2-factorization in which each 2-factor contains k r-cycles and an $(n - kr)$-cycle for the following values of n:*

1. $2(rm-1)+1 \leq n \leq \left\lfloor \frac{m}{k} \right\rfloor (rm-1)+2$, *if rm is odd, and*
2. $2(rm-2)+1 \leq n \leq \left\lfloor \frac{m}{k} \right\rfloor (rm-2)+2$, *if rm is even.*

Lemma 4 *Let $r \geq 3$, $k \geq 1$ and $n = 2(2kr-2)+2$ be integers with $(r, k) \notin \{(3,1), (3,2)\}$. Then K_n^* has a 2-factorization in which each 2-factor contains k r-cycles and an $(n - kr)$-cycle.*

Figure 1 shows the values of n which are obtained with the above lemmas for small k and r.

We also have the further result:

Theorem 5 *Let $3 \leq r \leq 9$, $n \geq r+3$ be integers. Then OP$(r, n-r)$ has a solution except for the cases OP(3,3) and OP(4,5). Let $3 \leq r \leq 4$, $n \geq 2r+3$ be integers. Then OP$(r, r, n-2r)$ has a solution except for the case OP$(3,3,5)$.*

Many of these cases follow from Lemmas 3 and 4, or are proved in [12, 13, 14]. We have solutions to all the remaining cases. In the final section we describe our method for obtaining these solutions and, as an example, give solutions to all the outstanding cases of OP$(5, n-5)$.

We must show that Theorem 2 is a consequence of Lemmas 3 and 4. For each valid pair (r, k), Lemma 3 provides a series of intervals such that if an integer n lies in one of these intervals a solution to OP$(r^k, n-kr)$ can be found. For example, consider when r is even: the intervals are given by the last line of

r	$k=1$	$k=2$	$k=3$	$k=4$
3	$17 \leq n \leq 26$ $n \geq 29$	$n = 29, 30$ $n \geq 33$	$n = 34, 41, 42, 45,$ 46 $n \geq 53$	$n = 46, 53, 54, 57, 58,$ $65, 66$ $n \geq 69$
4	$n = 14$ $n \geq 21$	$n = 30, 37, 38$ $n \geq 45$	$n = 46, 53, 54, 61,$ 62 $n \geq 69$	$n = 62, 69, 70, 77, 78,$ $85, 86$ $n \geq 93$
5	$n = 18$ $n \geq 29$	$n = 38, 49, 50$ $n \geq 57$	$n = 58, 69, 70, 77,$ 78 $n \geq 89$	$n = 78, 89, 90, 97, 98,$ $109, 110$ $n \geq 117$
6	$n = 22$ $n \geq 33$	$n = 46, 57, 58$ $n \geq 69$	$n = 70, 81, 82, 93,$ 94 $n \geq 105$	$n = 94, 105, 106, 117,$ $118, 129, 130$ $n \geq 141$
7	$n = 26$ $n \geq 41$	$n = 54, 69, 70$ $n \geq 81$	$n = 82, 97, 98,$ $109, 110$ $n \geq 125$	$n = 110, 125, 126, 137,$ $138, 153, 154$ $n \geq 165$
8	$n = 30$ $n \geq 45$	$n = 62, 77, 78$ $n \geq 93$	$n = 94, 109, 110,$ $125, 126$ $n \geq 141$	$n = 126, 141, 142, 157,$ $158, 173, 174$ $n \geq 189$
9	$n = 34$ $n \geq 53$	$n = 70, 89, 90$ $n \geq 105$	$n = 106, 125, 126,$ $141, 142$ $n \geq 161$	$n = 142, 161, 162, 177,$ $178, 197, 198$ $n \geq 213$

Figure 1: Some cases of $\mathrm{OP}(r^k, n-kr)$ solved by Lemmas 3 and 4.

the lemma. The intervals abut or overlap if the upper bound of one interval, $\left\lfloor \frac{m}{k} \right\rfloor (rm-2)+2$, is greater than or equal to the lower bound of the next, $2(r(m+1)-2)+1 = 2rm+2r-3$; that is, if $m \geq 3k$. Therefore for all values of n greater than $2(r(3k)-2)+1 = 6kr-3$, $\mathrm{OP}(r^k, n-kr)$ has a solution. For $2k+1 \leq m \leq 3k-1$, as $\left\lfloor \frac{m}{k} \right\rfloor = 2$ we can find solutions if $2(rm-2)+1 \leq n \leq 2(rm-2)+2$; that is, if $n \in \{2rm-3, 2rm-2 \mid m = 2k+1, \ldots, 3k-1\}$, which is the set described in the first part of Theorem 2. The next two parts of the theorem can also be seen to follow from Lemma 3 using similar arguments. The final part follows from Lemma 4.

We shall prove Lemma 3 by showing that an edge-colouring of K_{rm} that satisfies certain conditions can be extended to obtain a 2-factorization of K_n^*. The method of extending edge-colourings is introduced in the next section. In Section 3 we present some results which we shall use to find the initial edge-colouring of K_{rm}.

First, some further definitions and notation are presented. An r-cycle is denoted $[v_1, \ldots, v_r]$. An edge joining vertices v_i and v_j is denoted (v_i, v_j), and a sequence of adjacent edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$ will be abbreviated $(v_1, v_2, \ldots, v_n)$. The degree of a vertex v in a graph G is denoted $d_G(v)$. The maximum degree of the vertices in a graph G is denoted $\Delta(G)$. An *edge-colouring* of a graph G is a function $f\colon E(G) \to C$, where C is a set of colours. If G is an edge-coloured graph, then $G(c_i)$ is the subgraph induced by edges coloured c_i. A cycle is c_i-coloured if all its edges are coloured c_i and is (c_i, c_j)-coloured if all its edges are coloured either c_i or c_j and it contains at least one edge of each colour.

2 Extending Edge-Colourings

Theorem 6 is an adaptation of the outline/amalgamation result obtained for Hamiltonian decompositions of complete graphs in [8]. It also generalizes results in [5, 9]. It concerns a K_m edge-coloured with t colours, $c_1, \ldots, c_t$, and a set of associated parameters, $(s_1, \ldots, s_t)$, where each $s_i \in \{1, 2\}$ and $\sum_{i=1}^{t} s_i = n - 1$. We give necessary and sufficient conditions for such an edge-colouring to be extendible to an edge-colouring of K_n in which $K_n(c_i)$ is an s_i-factor, and if $s_i = 2$ $K_n(c_i)$ contains just one more cycle than $K_m(c_i)$. The case in which each $s_i = 2$ was proved in [9].

Theorem 6 *Let m and n be integers, $1 \le m < n$. Let $(s_1, \ldots, s_t)$, $s_i \in \{1, 2\}$, $1 \le i \le t$, be a composition of $n - 1$. Let K_m be edge-coloured with t colours $c_1, \ldots, c_t$. Let f_i be the number of edges coloured c_i. This colouring can be extended to an edge-colouring of K_n in which $K_n(c_i)$ is an s_i-factor, $1 \le i \le t$, and when $s_i = 2$ $K_n(c_i)$ contains exactly one more cycle than $K_m(c_i)$ if and only if*

$$\begin{array}{lll} \text{(A1)} & f_i \ge s_i\left(m - \dfrac{n}{2}\right) & (1 \le i \le t), \\ \text{(A2)} & s_i n \textit{ is even} & (1 \le i \le t), \\ \text{(A3)} & \Delta(K_m(c_i)) \le s_i & (1 \le i \le t). \end{array}$$

Proof of necessity in Theorem 6: K_m contains f_i edges coloured c_i. Each of the $(n - m)$ further vertices is incident with s_i edges coloured c_i. In K_n there must be exactly $s_i n/2$ edges coloured c_i. Hence

$$f_i + s_i(n - m) \;\; \ge \;\; \frac{s_i n}{2}.$$

Rearranging, (A1) is obtained.

An s_i-factor of K_n has $s_i n/2$ edges, and each vertex is incident with s_i edges. Hence (A2) and (A3) are necessary. $\square$

Before we can prove sufficiency in Theorem 6 we require a result concerning edge-colourings of bipartite multigraphs.

Given an edge-colouring of a loopless multigraph G with colours $c_1, \dots, c_n$, for each $v \in V(G)$, let $C_i(v)$ be the set of edges incident with v of colour c_i. An edge-colouring is *equitable* if, for all $v \in V(G)$,

$$\max_{1 \le i < j \le n} ||C_i(v)| - |C_j(v)|| \le 1.$$

The following result is due to de Werra [16, 17, 18]. A straightforward proof can be found in [4].

Proposition 7 *(de Werra) For each positive integer k, any finite bipartite multigraph has an equitable edge-colouring with k colours.*

Proof of sufficiency in Theorem 6: The greater part of this proof is devoted to showing that if $m < n-1$, the edge-colouring of K_m can be extended to an edge-colouring of K_{m+1} in such a way that (A1), (A2) and (A3) remain satisfied, with m replaced by $m+1$, and, if $s_i = 2$, $K_{m+1}(c_i)$ contains no more cycles than $K_m(c_i)$. By repeating this argument a finite number of times an edge-colouring of K_{n-1} that satisfies (A1), (A2) and (A3), with m replaced by $n-1$, can be found. We first show that such a colouring of K_{n-1} can be used to find the required factorization of K_n.

First note that each vertex in K_{n-1} has degree $n-2$, that $(s_1, \dots, s_t)$ is a composition of $n-1$, and that (A3) is satisfied. Therefore the edge-coloured K_{n-1} has the property (P): Each vertex is incident with s_i edges of colour c_i for $t-1$ values of i, and with $s_i - 1$ edges of colour c_i for one value of i.

From K_{n-1} we obtain K_n by adding a vertex v_n and joining it by one edge to each existing vertex. With $m = n-1$, (A1) becomes $f_i \ge s_i(n/2-1)$ $(1 \le i \le t)$. Therefore

$$\sum_{i=1}^{t} f_i \ge \frac{(n-1)(n-2)}{2},$$

and, as there are $(n-1)(n-2)/2$ edges in K_{n-1}, each f_i must be exactly $s_i(n/2-1)$. If $s_i = 1$, then there are $n/2-1$ edges coloured c_i. Since, by (A3), these edges are independent, there is just one vertex not incident with an edge coloured c_i in K_{n-1}. The edge joining this vertex to v_n is coloured c_i. Thus $K_n(c_i)$ is a 1-factor. If $s_i = 2$, then, by (P), each vertex must be incident with at least one edge coloured c_i. Thus the $n-2$ edges coloured c_i cannot all lie in cycles; there must be exactly one path of edges coloured c_i. The vertices at either end of this path are joined to v_n by edges coloured c_i. Hence $K_n(c_i)$ is a 2-factor containing one more cycle than $K_{n-1}(c_i)$. We have described how to colour $n-1$ edges incident to v_n, and, by (P), these edges must be distinct.

Now consider the case when $m < n-1$. Construct a bipartite multigraph B with vertex sets $\{c'_1, \dots, c'_t\}$ and $\{v'_1, \dots, v'_m\}$. For $1 \le i \le t$, $1 \le j \le m$, join

c_i' to v_j' by x edges, $x \in \{0,1,2\}$, if there are $(s_i - x)$ edges of colour c_i incident with v_j in K_m. (Consider the bipartite graph with the same vertex sets as B in which each c_i' is joined to each v_i' by s_i edges. If for each edge (v_j, v_k) of colour c_i in K_m we delete $c_i'v_j'$ and $c_i'v_k'$, then B will be obtained.) Notice that

$$d_B(v_j') = \left(\sum_{i=1}^{p} s_i\right) - (m-1) = n - m \qquad (1 \leq j \leq m).$$

Also notice that $d_B(c_i') = s_i m - 2f_i$. Then, considering (A1), we find that

$$d_B(c_i') \leq s_i m - 2s_i(m - n/2) = s_i(n-m) \qquad (1 \leq i \leq t).$$

Let B be given an equitable edge-colouring with $n-m$ colours, $\kappa_1, \ldots, \kappa_{n-m}$. Let B^* be the multigraph induced by the edges coloured κ_1 and κ_2. Notice that

$$\begin{array}{ll} d_{B^*}(v_j') = 2 & (1 \leq j \leq m), \\ d_{B^*}(c_i') \leq 2s_i & (1 \leq i \leq t), \\ |E(B^*)| = 2m. & \end{array}$$

For $1 \leq i \leq t$, if in K_m two vertices v_j and v_k lie at either end of a path of edges coloured c_i, where $s_i = 2$, and in B^* v_j' and v_k' are both adjacent to c_i', then v_j' and v_k' form an *i-pair*.

From B^* a further bipartite multigraph B^+ is constructed. If $s_i = 2$, vertex c_i' is split into two vertices, c_{i1}' and c_{i2}'. For each edge (c_i', v_j') in B^*, if $s_i = 1$, then there is an edge (c_i', v_j') in B^+; if $s_i = 2$, then there is an edge (c_{il}', v_j'), for some $l \in \{1,2\}$, in B^+. Furthermore, these latter edges are constructed such that, for $1 \leq i \leq t$, $1 \leq l \leq 2$, $d_{B^+}(c_{il}') \leq 2$, and if v_j' and v_k' are an i-pair, then in B^+ there are edges (c_{il}', v_j') and (c_{il}', v_k') for some $l \in \{1,2\}$.

B^+ is given an equitable edge-colouring with two colours, α and β. This edge-colouring is transferred to B^*. Let $B^*(\alpha)$ be the subgraph of B^* induced by edges coloured α. Notice that

$$d_{B^*(\alpha)}(v_j') = 1 \qquad (1 \leq j \leq m), \tag{1}$$

$$d_{B^*(\alpha)}(c_i') \leq s_i \qquad (1 \leq i \leq t), \tag{2}$$

$$|E(B^*(\alpha))| = m,$$

and, if v_j' and v_k' are an i-pair, then exactly one of the edges (c_i', v_j') and (c_i', v_k') is in $B^*(\alpha)$.

The edge-colouring of K_m can be extended to an edge-colouring of K_{m+1} by adding a vertex v_{m+1} which is joined to each existing vertex by one edge. For $1 \leq j \leq m$, if (c_i', v_j') is an edge of $B^*(\alpha)$, then $v_{m+1}v_j$ is coloured c_i. By (1), the colour of each new edge is precisely determined, and, by (2), v_{m+1} is incident with no more than s_i edges of colour c_i. The construction of B ensures that no other vertex is incident with more than s_i edges of colour c_i in K_{m+1}, so (A3) remains satisfied. For $1 \leq i \leq t$, $K_{m+1}(c_i)$ contains no more cycles than $K_m(c_i)$ as we ensured, by the creation of i-pairs, that if there is a path of edges coloured c_i, then v_{m+1} cannot be joined by edges coloured c_i to both ends of this path.

We must check that (A1) remains satisfied with m replaced by $m+1$.

If $s_i = 1$, and initially $f_i \geq m - n/2 + 1$, then (A1) will remain satisfied; if initially $f_i = m - n/2$, then $d_B(c_i') = n - m$, $d_{B^*}(c_i') = 2$ and $d_{B^*(\alpha)}(c_i') = 1$, so one further edge in K_{m+1} is coloured c_i and (A1) is still satisfied.

If $s_i = 2$, and initially $f_i \geq 2(m - n/2) + 2$, then (A1) will remain satisfied; if initially $f_i = 2(m-n/2)+1$, $d_B(c_i') = 2(n-m)-2$, $d_{B^*}(c_i') \geq 2$ and $d_{B^*(\alpha)}(c_i') \geq 1$, so at least one further edge in K_{m+1} is coloured c_i; if initially $f_i = 2(m - n/2)$, $d_B(c_i') = 2(n - m)$, $d_{B^*}(c_i') = 4$ and $d_{B^*(\alpha)}(c_i') = 2$, so two further edges in K_{m+1} are coloured c_i. In both of the latter two cases (A1) remains satisfied. $\square$

3 Resolvable Cycle Systems

In the next section Lemma 3 will be proved by edge-colouring a graph and applying Theorem 6. In this section we introduce some results that will help us find the initial edge-colouring. A fuller description of these results can be found in [15].

An *r-cycle system* of order rm is an edge-disjoint collection of r-cycles that partitions K_{rm}. A set of m cycles within a system forms a *parallel class* if each vertex of K_{rm} is incident with exactly one of the cycles. A cycle system is *resolvable* if the cycles can be partitioned into parallel classes. Clearly each parallel class is a 2-factor of K_{rm} comprising m r-cycles, and therefore rm, and hence also r and m, must be odd. Alspach, Schellenberg, Stinson and Wagner [2] have proved the following result.

Proposition 8 *Let $r \geq 3$ and $m \geq 1$ be positive odd integers. Then there exists a resolvable r-cycle system of K_{rm}.*

The analogous structure for even rm is a *nearly resolvable r-cycle system* of order rm. This is a partition of K_{rm} less a 1-factor into 2-factors each comprising m r-cycles. The following result was proved for all cases except $m = 4$ by Alspach, Schellenberg, Stinson and Wagner [2]; the remaining case was proved by Hoffman and Schellenberg [10].

Proposition 9 *Let $r \geq 3$ and $m \geq 1$ be positive integers. Then there exists a nearly resolvable r-cycle system of K_{rm} if and only if rm is even and $(r, m) \notin \{(3, 2), (3, 4)\}$.*

4 Proof of Lemma 3

Let p and q be the number of 2-factors in the 2-factorizations of K^*_{rm} and K^*_n respectively. Then

$$p = \left\lfloor \frac{rm-1}{2} \right\rfloor, \qquad q = \left\lfloor \frac{n-1}{2} \right\rfloor.$$

We know from Propositions 8 and 9 that K^*_{rm} has a 2-factorization in which the 2-factors $F_1, \ldots, F_p$ each consist of m r-cycles. Very roughly, the idea of the proof is to separate out q sets of k r-cycles, each such set lying in one of $F_1, \ldots, F_p$, and to colour these sets of cycles with the colours $c_1, \ldots, c_q$. Then the remaining edges of K^*_{rm} are coloured using the colours $c_1, \ldots, c_q$, but without creating any further cycles in these colours. Theorem 6 is used to extend each colour class to a 2-factor of K^*_n, where the 2-factor contains k r-cycles and an $(n-kr)$-cycle, and where the set of all such 2-factors forms a 2-factorization of K^*_n.

Suppose that rm and n are both even. Our assumption is that

$$2(rm-2)+1 \leq n \leq \left\lfloor \frac{m}{k} \right\rfloor (rm-2)+2.$$

Since n is even, this is equivalent to

$$2(rm-2)+2 \leq n \leq \left\lfloor \frac{m}{k} \right\rfloor (rm-2)+2,$$

which in turn is equivalent to

$$2\left(\frac{rm-2}{2}\right) \leq \frac{n-2}{2} \leq \left\lfloor \frac{m}{k} \right\rfloor \left(\frac{rm-2}{2}\right).$$

That is

$$2p \leq q \leq \left\lfloor \frac{m}{k} \right\rfloor p.$$

Very similar arguments show that this inequality holds in all the other cases as well.

In the edge-colouring of K^*_{rm} which we shall obtain from the 2-factorization $F_1, \ldots, F_p$ of K^*_{rm}, each colour c_i, $1 \leq i \leq q$, will be used on the edges of k r-cycles all belonging to the same 2-factor. Let us show that this is possible. Since each of the p 2-factors of K^*_{rm} contains m r-cycles, it is possible to select $\left\lfloor \frac{m}{k} \right\rfloor$ sets of k r-cycles from any 2-factor of K^*_{rm}, and so it is possible to pick out altogether $\left\lfloor \frac{m}{k} \right\rfloor p$ sets of k r-cycles, each of the cycles in each set lying in the same 2-factor of K^*_{rm}. Thus, since $q \leq \left\lfloor \frac{m}{k} \right\rfloor p$, it is possible to colour the edges of q sets of k r-cycles so that the edges in each r-cycle of each set receive the same colour, and no vertex has more than two edges of any colour incident with it. Therefore, for

$1 \le i \le q$, the colour c_i is used on the edges of k r-cycles of the 2-factor F_j if $i \equiv j \bmod p$.

Since $q \ge 2p$ it follows that, in particular, the two colours c_j and c_{p+j} are used on the edges of the 2-factor F_j of K^*_{rm}. If rm and n are both even, then we actually consider K_{rm} with a further colour, c_{q+1}, used on a 1-factor. In the cases when $rm \equiv n \bmod 2$ and when n is even and rm is odd, then, from any given 2-factor, the selection of the sets of k r-cycles to be coloured the same can be made arbitrarily. Any remaining cycles in a 2-factor F_j can be (c_j, c_{p+j})-coloured.

We now show that we can apply Theorem 6 to obtain the required 2-factorization of K^*_n. We consider the four cases separately.

Case 1: $n \equiv rm \equiv 1 \bmod 2$.

Clearly (A2) and (A3) are satisfied. To verify (A1), that $f_i \ge 2rm - n$ $(1 \le i \le q)$, we note that, since $n \ge 2(rm-1)+1$, (A1) follows from $f_i \ge 1$. This is clearly true since each colour c_i is used on all the edges of some r-cycle.

Case 2: $n \equiv rm \equiv 0 \bmod 2$.

In this case we now have an edge-colouring of K_{rm} with a further colour, c_{q+1}, occurring on a 1-factor. We need to extend this to an edge-colouring of K_n in such a way that c_{q+1} occurs on a 1-factor of K_n, and the other colours each form a 2-factor with k r-cycles and an $(n-kr)$-cycle. (A2) and (A3) are clearly satisfied. For (A1) we have to show that

$$\begin{aligned} &f_i \ge 2rm - n \qquad (1 \le i \le q), \text{ and} \\ &f_{q+1} \ge rm - n/2. \end{aligned}$$

Since $n \ge 2(rm-2)+2$ and n is even, these inequalities follow from the inequalities $f_i \ge 2$, $1 \le i \le q$, (which is true since c_i occurs on all the edges of at least one r-cycle) and $f_{q+1} = rm/2 = p+1 \ge 1$.

(Notice that in Lemma 4 we can let $m = 2k$ and then we also have $n \equiv rm \equiv 0 \bmod 2$. The proof is essentially the same as for Case 2 of Lemma 3. We can put $p = rk-1$, $q = 2(rk-1)$, and then the argument follows easily.)

Case 3: $n \equiv rm + 1 \equiv 0 \bmod 2$.

In this case the 2-factorization of K^*_n that we require is equivalent to a factorization of K_n with one further colour, say c_{q+1}, occurring on a 1-factor. Since $n \ge 2(rm-1)+1$ and n is even, we have $n \ge 2rm$, so (A1) reduces to the vacuous condition $f_i \ge 0$ $(1 \le i \le q+1)$.

Case 4: $n \equiv rm + 1 \equiv 1 \bmod 2$.

In this case the 2-factorization of K^*_{rm} is equivalent to a decomposition of K^*_{rm} into $p = \frac{1}{2}(rm-2)$ 2-factors $F_1, \ldots, F_p$ and a 1-factor, say F_{p+1}. Also in this case $K^*_n = K_n$. As before we shall select q sets of k r-cycles, each set lying in exactly one of $F_1, \ldots, F_p$, where, if $i + sp \le q$, one set of k r-cycles in F_i is coloured c_{i+sp}. The remaining r-cycles in F_i are (c_i, c_{i+p})-coloured. The extra difficulty in this case is that the edges of the 1-factor F_{p+1} need to be coloured with the colours $c_1, \ldots, c_q$ so that no vertex has more than two edges of any colour incident with it and no further cycles of any colour are created. In order to carry out this task, it is convenient not to select the k-sets of r-cycles arbitrarily, but instead to proceed more cautiously. We shall colour two edges of F_{p+1} with the colours used on the cycles of F_1, and, for $2 \le i \le p$, we shall colour one edge of F_{p+1} with the colour used on a cycle of F_i. Since F_{p+1} has $rm/2 = \frac{1}{2}(rm-2) + 1 = p + 1$ edges, this will deal with all the edges of F_{p+1}.

Let S be a set of k r-cycles in F_1. At most kr edges of F_{p+1} are incident with vertices in S, so at least $\frac{1}{2}rm - kr \ge \frac{1}{2}(2k+1)r - kr = \frac{1}{2}r$ edges of F_{p+1} are not incident with vertices in S. Therefore if $r \ge 4$, there are at least two edges of F_{p+1} that are not incident with vertices in S. If $r = 3$ then m is even, so $m \ge 2k+2$, and the same conclusion may be drawn. Let e^* and e^+ be edges of F_{p+1} that are not incident with cycles in S.

First suppose that $k = 1$, so that we require only one r-cycle of each colour. Then S contains just one r-cycle, say C_{1+p}. Let $e_0 = e^*$ and $e_1 = e^+$, let the remaining edges of F_{p+1} be $e_2, \ldots, e_p$, and let $e_i = (v_i, w_i)$, $0 \le i \le p$. Let C_1 be an r-cycle in F_1 with $|V(C_1) \cap \{v_0, w_0, v_1, w_1\}| \ge 1$. We colour C_1 with colour c_1, C_{1+p} with colour c_{1+p}, and e_0 and e_1 with c_{1+p} as well. For each j such that $1 + 2p \le 1 + jp \le q$, one r-cycle of F_1 will be coloured with c_{1+jp}. Let T be a set containing the remaining uncoloured cycles of F_1. These cycles must be (c_1, c_{1+p})-coloured. They may be incident with e_0 and e_1 so we must find a way to colour them so that no c_{1+p}-coloured cycle containing e_0 or e_1 is formed, no vertex is incident with three c_{1+p}-coloured edges, and c_1 and c_{1+p} are each used at least once on each of the cycles in T. We can assume that $v_1 \in C_1$. If v_0 is in a cycle in T then we colour the two edges of that cycle incident with v_0 with c_1. Therefore neither e_0 nor e_1 can be in a c_{1+p}-coloured cycle. If w_0 is in a cycle in T then we colour one of the edges of that cycle that is incident with w_0 with c_1. Note that we have not yet coloured all the edges of any cycle in T since we have only coloured three edges and if v_0 and w_0 are in the same cycle it must contain at least four edges as v_0 and w_0 cannot be adjacent. If w_1 is in a cycle in T then we colour one of the edges of that cycle that is incident with w_1 with c_1 if we have not done so already (w_1 may be adjacent to v_0 or v_1). We have ensured that no vertex is incident with three c_{1+p}-coloured edges and we have still not coloured all the edges of any cycle in T since one of the edges incident with w_1 (if it is in a cycle in T) was left uncoloured. We complete the colouring of the cycles in T ensuring that c_1 and c_{1+p} are each used on at least one edge of each cycle. For

$2 \leq i \leq p$, the cycle in F_i containing v_i is coloured c_i, and e_i is coloured c_{i+p}. If w_i is in a (c_i, c_{i+p})-coloured r-cycle, we make sure that one edge incident with w_i is coloured c_i.

Now consider the case when $k \geq 2$. Let $e_0 \in F_{p+1}$, $e_0 = (v_0, w_0)$. We can choose S, a set of at least two r-cycles now, so that the cycles in S contain v_0 and w_0. Colour the edges of the cycles in S with c_1, and colour e_0 with c_{1+p}. Recall that an edge $e^* \in F_{p+1}$ is disjoint from S. Let $e_1 = e^* = (v_1, w_1)$. Colour e_1 with colour c_1. The r-cycles incident with (v_1, w_1) form part of a set of k r-cycles that we colour c_{1+p}. Let the remaining edges of F_{p+1} be $e_2, \ldots, e_p$, where, for $2 \leq i \leq p$, $e_i = (v_i, w_i)$. For $2 \leq i \leq p$, v_i and w_i lie in a set of k r-cycles that we colour c_i, and we colour e_i with c_{i+p}. For $1 \leq i \leq p$, the remaining cycles of F_i are (c_i, c_{i+p})-coloured.

Clearly (A2) and (A3) are satisfied. (A1) is satisfied since $f_i \geq 3 = 2rm - (2(rm-2)+1)$, as each colour c_i is used on all the edges of some cycle. $\square$

5 Proof method for Theorem 5

In this section we describe the method we used to solve the remaining cases of OP$(r, n-r)$, $3 \leq r \leq 9$, and OP$(r, r, n-2r)$, $3 \leq r \leq 4$. We again use Theorem 6. As in the proof of the last section, if n is odd, then $s_i = 2$, $1 \leq i \leq t$; if n is even, then, for $1 \leq i \leq t-1$, $s_i = 2$, and s_t=1. We choose a value of m and colour the edges of K_m. If $s_i = 2$, $K_m(c_i)$ contains k r-cycles and no other cycles. Thus, if (A1), (A2) and (A3) are satisfied, then the edge-colouring is equivalent to a solution of OP$(r^k, n-kr)$. (A1) determines the minimum size of $K_m(c_i)$. (A2) and (A3) are clearly satisfied. We shall see that by recolouring only a few edges many solutions can be found quickly from one initial colouring.

We demonstrate this method by solving the remaining cases of OP$(5, n-5)$. Solutions are already known for OP$(5,3)$, OP$(5,5)$, OP$(5,7)$, OP$(5,9)$ and OP$(5,11)$ [3] and for OP$(5,13)$ and OP$(5, n-5)$ for $n \geq 29$ (Lemmas 3 and 4). It is known that OP$(5,4)$ has no solution [3]. We present solutions for the remaining cases. An edge-colouring of K_m is given by describing the subgraph induced by each colour. Recall that $[v_1, \ldots, v_r]$ is a cycle, and $(v_1, \ldots, v_r)$ is a sequence of adjacent edges where v_1 is not adjacent to v_r.

OP$(5,6)$: $m = 9$; by (A1), we require $f_i \geq 7$, $1 \leq i \leq 5$.

$K_9(c_1) = [1,2,3,4,5]$ $(6,7,8)$
$K_9(c_2) = [1,3,5,6,9]$ $(7,2,8)$
$K_9(c_3) = [1,4,2,9,8]$ $(6,3,7)$
$K_9(c_4) = [3,8,6,4,9]$ $(1,7,5,2)$
$K_9(c_5) = [4,7,9,5,8]$ $(1,6,2)$

OP$(5,8)$: $m=9$; by (A1), we require $f_i \geq 5$, $1 \leq i \leq 6$. The solution is as above, except that five edges are recoloured to give

$$K_9(c_6) = [2,5,7,3,6]$$

OP$(5,10)$: $m=13$; by (A1), we require $f_i \geq 11$, $1 \leq i \leq 7$.

$$\begin{aligned}
K_{13}(c_1) &= [1,2,3,4,5]\ (6,7,8)\ (9,10,11,12,13)\\
K_{13}(c_2) &= [1,3,5,2,4]\ (7,12,8,13,11,9,6,10)\\
K_{13}(c_3) &= [1,6,8,5,10]\ (3,13,7,9,2,12)\ (4,11)\\
K_{13}(c_4) &= [1,7,2,6,11]\ (4,8,3,9,13,5)\ (10,12)\\
K_{13}(c_5) &= [1,8,10,4,9]\ (2,13)\ (3,6,12,5,11,7)\\
K_{13}(c_6) &= [1,12,3,10,13]\ (2,8,11)\ (6,4,7,5,9)\\
K_{13}(c_7) &= [2,10,7,3,11]\ (5,6,13,4,12,9,8)
\end{aligned}$$

For OP$(5,12)$, OP$(5,14)$, OP$(5,16)$, OP$(5,18)$, OP$(5,20)$ and OP$(5,22)$, the solution is as above, except that we recolour a number of edges for each new solution. For OP$(5,12)$ we require that each $f_i \geq 9$, for OP$(5,14)$ we require that each $f_i \geq 7$ and for the remaining solutions it is sufficient that each $f_i \geq 5$.

$$\begin{aligned}
&\mathrm{OP}(5,12)\text{:} && K_{13}(c_8) = [2,8,3,6,13]\ (4,7,9,10,11)\\
&\mathrm{OP}(5,14)\text{:} && K_{13}(c_9) = [2,9,5,7,12]\ (8,13,11)\\
&\mathrm{OP}(5,16)\text{:} && K_{13}(c_{10}) = [3,9,7,11,13]\\
&\mathrm{OP}(5,18)\text{:} && K_{13}(c_{11}) = [4,7,6,12,11]\\
&\mathrm{OP}(5,20)\text{:} && K_{13}(c_{12}) = [4,8,11,9,12]\\
&\mathrm{OP}(5,22)\text{:} && K_{13}(c_{13}) = [5,11,10,12,13]
\end{aligned}$$

For OP$(5,15)$, OP$(5,17)$, OP$(5,19)$, OP$(5,21)$ and OP$(5,23)$, we can adapt the above solutions. The solution for OP$(5,2k-5)$ is the same as for OP$(5,2k-6)$, except that three edges are recoloured (a sufficient number to satisfy (A1)) to give

$$K_{13}(c_k) = (4,13)\ (5,12)\ (6,9)$$

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